In order to determine thermodynamic properties of free fermions one typically has to evaluate integrals of the type

$$I\left(T\right)=\int_{-\infty}^{\infty}H\left(\omega\right)f\left(\omega\right)d\omega \tag{1}$$

with $f\left(\omega\right)=\left(e^{\beta\left(\omega-\mu\right)}+1\right)^{-1}$ and $H\left(\omega\right)$ some function which is assumed to vanish for $\omega\rightarrow-\infty$. In case of the particle number $H\left(\omega\right)=N\rho_{0}\left(\omega\right)$, whereas in case of the internal energy $H\left(\omega\right)=\omega N\rho_{0}\left(\omega\right)$.

We introduce

$$K\left(\omega\right)=\int_{-\infty}^{\omega}H\left(\omega'\right)d\omega' \tag{2}$$

i.e. $H\left(\omega\right)=\frac{dK\left(\omega\right)}{d\omega}$ and get

$$I=-\int_{-\infty}^{\infty}K\left(\omega\right)\frac{\partial f\left(\omega\right)}{\partial\omega}d\omega. \tag{3}$$

At low temperatures $-\frac{\partial f\left(\omega\right)}{\partial\omega}$ is sharply peaked at the Fermi function such that only $K\left(\omega\right)$ close to $\omega=\mu$ contributes and we can expand

$$K\left(\omega\right)=K\left(\mu\right)+K'\left(\mu\right)\left(\omega-\mu\right)+\frac{1}{2}K''\left(\mu\right)\left(\omega-\mu\right)^{2}+... \tag{4}$$

Inserting this gives for the linear term zero since $\frac{\partial f\left(\omega\right)}{\partial\omega}$ is even with respect to μ. Due to $-\int_{-\infty}^{\infty}\frac{\partial f\left(\omega\right)}{\partial\omega}d\omega=1$ follows that

$$\begin{aligned}I &= K\left(\mu\right)-\frac{1}{2}K''\left(\mu\right)\int_{-\infty}^{\infty}\left(\omega-\mu\right)^{2}\frac{\partial f\left(\omega\right)}{\partial\omega}d\omega && (5)\\ &= K\left(\mu\right)-\frac{1}{2}K''\left(\mu\right)\left(k_{B}T\right)^{2}\int_{-\infty}^{\infty}x^{2}\frac{d}{dx}\frac{1}{e^{x}+1}dx && (6)\end{aligned}$$

The last integral can be done as well: $\int_{-\infty}^{\infty}x^{2}\frac{d}{dx}\frac{1}{e^{x}+1}dx=-\frac{\pi^{2}}{3}$ such that

$$I=\int_{-\infty}^{\mu\left(T\right)}H\left(\omega\right)d\omega+\frac{\pi^{2}}{6}H'\left(E_{F}\right)\left(k_{B}T\right)^{2}. \tag{7}$$

In the last step we have used the fact that at lowest order in temperature $\mu\simeq E_{F}$ and that $H\left(\omega\right)=K'\left(\omega\right)$, i.e. $H'\left(\omega\right)=K''\left(\omega\right)$.