Feldtheorie der Kondensierten Materie, RG und Quantenkritikalität SS 2013

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1. Matsubara summation

Show that for $a > 0$ it holds that

$$T \sum_{\omega_n} \frac{1}{\omega_n^2 + a^2} - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega^2 + a^2} = \frac{1}{a} \frac{1}{e^{a/T} - 1},$$

where $\omega_n = 2\pi T n$ are bosonic Matsubara frequencies.

2. Statics at nonzero temperatures in ϕ^4-theory

Consider the N-component ϕ^4 field theory in d spatial dimensions defined by the imaginary time path integral

$$Z = \int \mathcal{D}[\phi_\alpha(x,\tau)] \exp(-S)$$

with action

$$S = \int d^d x \int_0^{1/T} d\tau \left\{ \frac{1}{2} \left[(\partial_\tau \phi_\alpha)^2 + c^2 (\nabla_x \phi_\alpha)^2 + r \phi_\alpha^2(x,\tau) \right] + \frac{u}{4!} (\phi_\alpha^2(x,\tau))^2 \right\}.$$

(a) Set $u = 0$ and compute the susceptibility

$$\chi_{\alpha\beta}(k,\omega_n) = \int_0^{1/T} d\tau \int d^d x \, \langle \phi_\alpha(x,\tau) \phi_\beta(0,0) \rangle \, e^{-i(kx - \omega_n \tau)}$$

in the O(N) symmetric region of the phase diagram. Show that it holds that $\chi_{\alpha\beta} = \chi \delta_{\alpha\beta}$.

(b) Calculate the lowest order corrections of $\chi(k,\omega_n)$ due to nonzero u. Identify the shift of the critical value of r, where the phase transition from the disordered to the ordered state occurs, from $r_c(u=0) = 0$ to $r_c(u > 0) \neq 0$ due to nonzero u. The critical value $r_c(u)$ is defined as the value where $\chi(0,0)$ shows a divergence, since the correlation length is then infinite $\xi \to \infty$.

(c) Write $\chi^{-1}(q,\omega_n)$ in terms of $s = r - r_c(u)$ and perform the Matsubara summation and momentum integration. Distinguish between $d < 3$ and $d > 3$, *i.e.*, dimensions $D = d + 1$ below and above the upper critical dimension $D_c = 4$. Show that for $d < 3$ the correction diverges close to the phase transition ($s \to 0$) and extract the universal (cutoff independent) scaling function $\Psi_D(x,y)$ by writing

$$\chi^{-1}(i\omega_n, k) = s \Psi_D\Big(\frac{\sqrt{\omega_n^2 + k^2}}{\sqrt{s}}, \frac{u}{s^{(4-D)/2}} \Big)$$

For $d > 3$ expand in small s and show that the correction remains finite, but depends on the momentum cutoff Λ.

(d) To find the correlation length in different crossover regimes at finite temperature close to the quantum critical point, derive an effective action S_{eff} for the static $\omega_n = 0$ mode only. For this, integrate over all nonzero Matsubara modes up to one loop similarly to the RG procedure (fast modes are all modes with $\omega_n > 0$, slow mode is only $\omega_n = 0$) to arrive at

$$S_{\text{eff}} = \frac{1}{T}\int d^d x \Big\{ \frac{1}{2}\big[(\nabla_x \phi_\alpha)^2 + R\phi_\alpha^2(x)\big] + \frac{U}{4!}(\phi_\alpha^2(x))^2 \Big\}$$

with

$$R = r + u\frac{N+2}{6} T \sum_{\omega_n \neq 0} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\omega_n^2 + k^2 + s} + r_c - r_c$$

$$U = u - u^2 \frac{N+8}{6} T \sum_{\omega_n \neq 0} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(\omega_n^2 + k^2 + s)^2} \, .$$

where r_c was calculated in part (b) and $r - r_c = s$ and $s = 0$ marks the location of the quantum phase transition.

(e) Use the first-order result $U = u$ in the following and calculate the shift of R_c due to nonzero U exactly as in part (b) and thus find the equivalent of s as $\mathcal{S} = R - R_c(U)$ for the effective action. Evaluate the Matsubara sums [use the result from number (1)] in the expression of R to arrive at

$$\mathcal{S} = s + u\frac{N+2}{6}\Big\{ \int \frac{d^d k}{(2\pi)^d}\Big(\frac{1}{\sqrt{k^2+s}} \frac{1}{e^{\sqrt{k^2+s}/T} - 1} - \frac{T}{k^2+s} + \frac{T}{k^2} \Big)$$
$$+ T\sum_{\omega_n} \int^{\Lambda} \frac{d^d k}{(2\pi)^d} \Big(\frac{1}{\omega_n^2 + k^2 + s} - \frac{1}{\omega_n^2 + k^2} + \frac{s}{(\omega_n^2+k^2)^4} - \frac{s}{(\omega_n^2+k^2)^4} \Big\} ,$$

where we have added and subtracted $\frac{s}{(\omega_n^2+k^2)^4}$ in the last integral to remove the ultraviolet divergence. Evaluate the remaining Matsubara summations, and write the result in the form

$$\mathcal{S} = s(1 + c_1 u \Lambda^{d-3}) + uT^{d-1}\frac{N+2}{6} G_d\Big(\frac{s}{T^2}\Big)$$

with non-universal constant c_1 and universal function $G_d(x)$ still containing a momentum integral $\int_0^\infty d^d k$. You have now separated cutoff Λ dependent terms and universal T dependence.

(f) We can then use the analogy between the effective action S_{eff} and S to conclude that

$$\chi^{-1}(k) = \mathcal{S}\Psi_d\Big(\frac{\sqrt{\omega_n^2 + k^2}}{\sqrt{\mathcal{S}}}, \frac{TU}{\mathcal{S}^{(4-d)/2}} \Big)$$

with known function $\Psi_d(x, y)$ from part (c) (evaluated in spatial dimension d here). Write $\chi^{-1}(k) = k^2 + \xi^{-2}$ and analyze the temperature behavior of the correlation length ξ in the three regimes (A): $s > 0, T \ll \sqrt{s}$, (B): $s > 0, \sqrt{s} \ll T \ll (s/u)^{1/(d-1)}$ and (C): $s > 0, (s/u)^{1/(d-1)} \ll T$ close to the quantum critical point on the disordered O(N) symmetric side of the phase diagram.

(g) Find the finite temperature transition temperature T_c from the condition $s < 0$ and $\xi^{-2} = 0$.