Übungen zur Theorie der Kondensierten Materie II SS 13

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1. Electron-electron collisions in and dephasing time (20 Punkte)

In the lectures weak localization correction to conductivity of disordered metals was discussed. In 1D and 2D this correction diverges in infrared. The divergence can be cut off by system size or external frequency. On the other hand, for non-interacting electrons, finite temperature can *not* cut the weak localization correction. The situation changes, if the electron-electron interaction is switched on. Finite phase breaking time $\tau_\phi(T)$ provides mass to Cooperon and cuts the divergence making the weak localization correction indeed weak at high enough temperatures. In this exercise we estimate $\tau_\phi(T)$.

<u>Electron-electron collision rate at $T = 0$</u>.

Let us consider electrons living in short-range correlated random potential with single-particle eigenstates $\psi_m(x)$. In the absence of interaction electrons fill those eigenstates independently. The many-particle ground state is obtained by filling all single-particle eigenstates with energies $\epsilon_m < 0$ (energies are counted from chemical potential). The simplest excited state is obtained by adding an electron into eigenstate $\psi_\alpha(x)$ with $\epsilon_\alpha > 0$. We denote such a state by $|e_\alpha\rangle$. We are interested in average decay rate of such states caused by some interaction $U(q, \omega)$. For states with $\epsilon_m < 0$ we denote by $|h_m\rangle$ the many-particle state with electron removed from the single-particle level ψ_m.

(a) Consider the simplest decay process $|e_\alpha\rangle \to |e_\beta h_\gamma e_\delta\rangle$. Use Fermi Golden Rule to show that the contribution of these processes to the life-time of the state $|e_\alpha\rangle$ is given by

$$\frac{1}{\tau_{ee}(\alpha)} = 2\pi \sum_{\beta\gamma\delta} |M_{\alpha\beta\gamma\delta}|^2 \delta(\epsilon_\alpha + \epsilon_\gamma - \epsilon_\beta - \epsilon_\delta) \tag{1}$$

$$M_{\alpha\beta\gamma\delta} = \int dr dr' U(r - r', \epsilon_\alpha - \epsilon_\beta) \psi_\alpha(r) \psi^*_\beta(r) \psi_\gamma(r') \psi^*_\delta(r') \tag{2}$$

The summation goes over states with $\epsilon_\gamma < 0$ and ϵ_β , $\epsilon_\delta > 0$. What is the diagram corresponding to this process?

(b) We define the decay rate averaged over all possible initial states $|e_\alpha\rangle$ at energy E and the realizations of the disorder

$$\frac{1}{\tau_{ee}(E)} = \left\langle \frac{1}{\nu V} \sum_\alpha \frac{1}{\tau_{ee}(\alpha)} \delta(E - \epsilon_\alpha) \right\rangle. \tag{3}$$

Show that

$$\frac{1}{\tau_{ee}(E)} = \frac{2\pi}{\nu V} \int_0^E d\omega \int_{-\omega}^0 d\epsilon \left\langle \sum_{\alpha\beta\gamma\delta} |M_{\alpha\beta\gamma\delta}|^2 \delta(E - \epsilon_\alpha) \delta(E - \omega - \epsilon_\beta) \delta(\epsilon - \epsilon_\gamma) \delta(\epsilon + \omega - \epsilon_\delta) \right\rangle \tag{4}$$

(c) Consider the impurity average in the equation above. Assume that the interaction range in real space is much larger than Fermi wavelength. Show that

$$\langle \ldots \rangle = \int \frac{dq}{(2\pi)^d} |U(q,\omega)|^2 S^2(\omega, q) \quad (5)$$

$$S(\omega, q) = \frac{1}{2\pi^2} \int d(r_1 - r_2) e^{-iq(r_1-r_2)} \mathrm{Re} \left\langle G^R_E(r_1, r_2) G^A_{E-\omega}(r_2, r_1) \right\rangle = \frac{\nu}{\pi} \mathrm{Re} \frac{1}{Dq^2 - i\omega} \quad (6)$$

Here $G^{R(A)}$ are *exact* Green functions in the disorder potential and d is dimensionality of the space.
Suggestion: Use that

$$\sum_\alpha \psi_\alpha(r) \psi^*_\alpha(r') \delta(E - \epsilon_\alpha) = \frac{1}{-2\pi i} \left[G^R_E(r, r') - G^A_E(r, r') \right] \quad (7)$$

(d) Assume now that $U(\omega, q)$ is the screened Coulomb interaction. Using (4), (5) and (6) estimate the relaxation rate $1/\tau_{ee}(E)$ for for dimensions 1, 2 and 3. Compare the result to the case of clean Fermi liquid.
Suggestion: The integral over q in (5) is determined by the region $Dq^2 \sim \omega$ where the dynamical screening is of minor importance.

Electron-electron collision rate at finite T and phase breaking time.

(e) Consider the decay rate for a state $|e_\alpha\rangle$ at finite temperature T. How should one modify equation (1) in this case?
Suggestion: Think about the factors describing the occupation of states β, γ and δ.

(f) Show that at finite T the scattering rate for a particle at energy $E \sim T$ can be estimated as

$$\frac{1}{\tau_{ee}(T)} \sim \frac{1}{\nu D^{d/2}} T \int_0^T d\omega \omega^{d/2-2} \quad (8)$$

Discuss the divergence of decay rate for $d \leq 2$. Can this divergence show up show up in physical quantities?

(g) Show that the processes with energy transfer $\omega < 1/\tau_\phi$ can not contribute to phase breaking. How should one modify equation (8) to estimate τ_ϕ? Estimate τ_ϕ in dimensions 1, 2 and 3.

2. Kubo formula for conductivity (10 Punkte)

Let us consider non-interacting Fermions in random potential.

(a) Derive the Kubo formula for the (averaged over disorder realizations) current response $j^\alpha(\omega, q)$ (with $\alpha = x, y, z$) to the externally applied vector potential $A^\alpha(\omega, q)$. Write the answer in terms of retarded and advanced Green functions.
Suggestion: the correct answer reads

$$j^\alpha(\omega, q) = \mathcal{K}^{\alpha\beta}(\omega, q) A^\alpha(\omega, q) \quad (9)$$

$$\mathcal{K}^{\alpha\beta}(\omega, q) = \frac{ie^2}{c} \int \frac{d\epsilon}{2\pi} d(r - r') e^{-iq(r-r')} \left[F_{\epsilon-\omega} \left\langle \widehat{v}^\alpha_r G^R_\epsilon(r, r') \widehat{v}^\beta_{r'} \left(G^R_{\epsilon-\omega}(r', r) - G^A_{\epsilon-\omega}(r', r) \right) \right\rangle \right.$$
$$\left. + F_\epsilon \left\langle \widehat{v}^\alpha_r \left(G^R_\epsilon(r, r') - G^A_\epsilon(r, r') \right) \widehat{v}^\beta_{r'} G^A_{\epsilon-\omega}(r', r) \right\rangle \right] - \frac{e^2}{2mc} n \delta^{\alpha\beta} \quad (10)$$

Here $F_\epsilon = \tanh \epsilon/2T$, n is the density of the fermions and $\widehat{v}^\alpha_r = -\frac{i}{m} \partial_{r^\alpha}$.

(b) Use the relation derived in (a) to show that the real part of the conductivity can be expressed as

$$\mathrm{Re}\sigma(\omega) = -\frac{1}{4\pi\omega}\int d\epsilon(n(\epsilon)-n(\epsilon-\omega))\int d(r-r') \\ \left\langle \widehat{v}_r^{\alpha}\left(G_\epsilon^R(r,r')-G_\epsilon^A(r,r')\right)\widehat{v}_{r'}^{\alpha}\left(G_{\epsilon-\omega}^R(r',r)-G_{\epsilon-\omega}^A(r',r)\right)\right\rangle \quad (11)$$