Karlsruher Institut für Technologie

Institut für Theorie der Kondensierten Materie

Moderne Theoretische Physik für Informatiker SS 2014

Prof. Dr. A. Shnirman

Dr. B. Narozhny

Blatt 3: Lösungen

Besprechung 06.05.2014

---

1. Lagrangefunktion:

(a) ‘Zwei Massen”

Koordinaten der Masse m_1

$$\vec{r}_1 = (x, 0, 0)$$

Kinetische Energie:

$$K_1 = \frac{1}{2} m_1 v_1^2, \qquad \vec{v}_1 = (\dot{x}, 0, 0) \qquad \Rightarrow \qquad K_1 = \frac{1}{2} m_1 \dot{x}^2$$

Koordinaten der Masse m_2

$$\vec{r}_2 = (x + l \sin\varphi, 0, -l\cos\varphi)$$

Kinetische Energie:

$$K_2 = \frac{m_2}{2} v_2^2, \quad \vec{v}_2 = (\dot{x} + l\dot{\varphi}\cos\varphi, 0, l\dot{\varphi}\sin\varphi) \quad \Rightarrow \quad K_2 = \frac{m_2}{2}\left[\dot{x}^2 + l^2\dot{\varphi}^2 + 2l\dot{x}\dot{\varphi}\cos\varphi\right]$$

Potentielle Energie

$$U = m_2 g z = -m_2 g l \cos\varphi$$

Die Lagrangefunktion

$$L = K_1 + K_2 - U$$

$$L = \frac{m_1 + m_2}{2}\dot{x}^2 + \frac{m_2}{2}\left[l^2\dot{\varphi}^2 + 2l\dot{x}\dot{\varphi}\cos\varphi\right] + m_2 g l \cos\varphi$$

Die Bewegungsgleichungen

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \Rightarrow \qquad (m_1 + m_2)\,\ddot{x} + m_2 l\left[\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi\right] = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0 \qquad \Rightarrow \qquad m_2 l^2\ddot{\varphi} + m_2 l\left[\ddot{x}\cos\varphi + g\sin\varphi\right] = 0$$

(b) "Zwei gekoppelte Pendel"

Koordinaten der Masse m_1

$$\vec{r}_1 = (l_1 \sin\varphi_1, l_1 \cos\varphi_1)$$

Kinetische Energie:

$$K_1 = \frac{m_1}{2} v_1^2, \qquad \vec{v}_1 = l_1 \dot{\varphi}_1 (\cos\varphi_1, -\sin\varphi_1) \qquad \Rightarrow \qquad K_1 = \frac{m_1}{2} l_1^2 \dot{\varphi}_1^2$$

Potentielle Energie

$$U_1 = -m_1 g l_1 \cos\varphi_1$$

Koordinaten der Masse m_2

$$\vec{r}_2 = (l_1 \sin\varphi_1 + l_2 \sin\varphi_2, l_1 \cos\varphi_1 + l_2 \cos\varphi_2)$$

Kinetische Energie:

$$K_2 = \frac{m_2}{2} v_2^2, \qquad \vec{v}_2 = (l_1 \dot{\varphi}_1 \cos\varphi_1 + l_2 \dot{\varphi}_2 \cos\varphi_2, -l_1 \dot{\varphi}_1 \sin\varphi_1 - l_2 \dot{\varphi}_2 \sin\varphi_2)$$

$$K_2 = \frac{m_2}{2} \left[l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + 2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \right]$$

Potentielle Energie

$$U_2 = -m_2 g \left(l_1 \cos\varphi_1 + l_2 \cos\varphi_2 \right)$$

Die Lagrangefunktion

$$L = K_1 + K_2 - U_1 - U_2$$

$$L = \frac{m_1 + m_2}{2} l_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2} l_2^2 \dot{\varphi}_2^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + (m_1 + m_2) g l_1 \cos\varphi_1 + m_2 g l_2 \cos\varphi_2$$

Die Bewegungsgleichungen

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_1} - \frac{\partial L}{\partial \varphi_1} = 0 \quad \Rightarrow$$

$$(m_1 + m_2) l_1^2 \ddot{\varphi}_1 + m_2 l_1 l_2 \left[\ddot{\varphi}_2 \cos(\varphi_1 - \varphi_2) + \dot{\varphi}_2^2 \sin(\varphi_1 - \varphi_2) \right]$$

$$+ (m_1 + m_2) g l_1 \sin\varphi_1 = 0$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}_2} - \frac{\partial L}{\partial \varphi_2} = 0 \quad \Rightarrow$$

$$m_2 l_2^2 \ddot{\varphi}_2 + m_2 l_1 l_2 \left[\ddot{\varphi}_1 \cos(\varphi_1 - \varphi_2) + \dot{\varphi}_1^2 \sin(\varphi_2 - \varphi_1) \right]$$

$$+ m_2 g l_2 \sin\varphi_2 = 0$$

2. **Erhaltungssätze:**

(a) “Impulse und Energien”

Impuls:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Energie

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

i)

$$p = \frac{\partial L}{\partial \dot{x}} = -\frac{3au\dot{x}^2}{2\sqrt{1-u\dot{x}^3}}$$

$$E = p\dot{x} - L = -\frac{3au\dot{x}^3}{2\sqrt{1-u\dot{x}^3}} - a\sqrt{1-u\dot{x}^3} + kx^2 \quad \Rightarrow$$

$$E = \frac{-3au\dot{x}^3 - 2a\left(1-u\dot{x}^3\right)}{2\sqrt{1-u\dot{x}^3}} + kx^2 = -\frac{2a + au\dot{x}^3}{2\sqrt{1-u\dot{x}^3}} + kx^2$$

ii)

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{m}{2}\dot{y}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{m}{2}\dot{x}$$

$$E = p_x\dot{x} + p_y\dot{y} - L = \left[m\dot{x} + \frac{m}{2}\dot{y}\right]\dot{x} + \left[m\dot{y} + \frac{m}{2}\dot{x}\right]\dot{y} - \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + \dot{x}\dot{y}\right] + cx^4 + dy^6$$

$$E = \frac{m}{2}\left[\dot{x}^2 + \dot{y}^2 + \dot{x}\dot{y}\right] + cx^4 + dy^6$$

iii)

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\left(l - a\varphi\right)^2 \dot{\varphi}$$

$$E = p_\varphi\dot{\varphi} - L = \frac{m}{2}\left(l - a\varphi\right)^2 \dot{\varphi}^2 + mg\left[a\cos\varphi - \left(l - a\varphi\right)\sin\varphi\right]$$

(b) “Zwei Massen”

Betrachten wir nur die Bewegung, wobei die masse m_1 auf der Ebene bleibt.

Kartesisches Koordinatensystem:

Koordinaten der Masse m_1

$$\vec{r}_1 = (x_1, y_1, 0)$$

Koordinaten der Masse m_2

$$\vec{r}_2 = (x_2, y_2, z_2)$$

"Die Lagrangefunktion"

$$L = \frac{m_1}{2}\left[\dot{x}_1^2 + \dot{y}_1^2\right] + \frac{m_2}{2}\left[\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right] + m_2 g z_2$$

Zusätzliche Zwangsbedingung

$$l = r_1 + r_2 \quad \Rightarrow \quad l = \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2 + z_2^2}$$

Masse m_1, Polarkoordinaten:

$$x_1 = r_1 \cos\phi_1, y_1 = r_1 \sin\phi_1$$

Masse m_2, sphärische Koordinate

$$x_2 = r_2 \cos\phi_2 \sin\theta_2, \quad y_2 = r_2 \sin\phi_2 \sin\theta_2, \quad z_2 = r_2 \cos\theta_2$$

Die Lagrangefunktion:

$$L = \frac{m_1 + m_2}{2}\dot{r}_1^2 + \frac{m_1}{2} r_1^2 \dot{\phi}_1^2 + \frac{m_2}{2}\left[(l - r_1)^2 \dot{\phi}_2^2 \sin^2\theta_2 + (l - r_1)^2 \dot{\theta}_2^2\right] + m_2 g (l - r_1)\cos\theta_2$$

i) Euler-Lagrange Gleichungen

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_1} - \frac{\partial L}{\partial r_1} = 0 \qquad \Rightarrow$$

$$(m_1 + m_2)\,\ddot{r}_1 - m_1 r_1 \dot{\phi}_1^2 - m_2(l - r_1)\left[\dot{\phi}_2^2 \sin^2\theta_2 + \dot{\theta}_2^2\right] + m_2 g \cos\theta_2 = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_1} - \frac{\partial L}{\partial \phi_1} = 0 \qquad \Rightarrow$$

$$m_1 r_1^2 \ddot{\phi}_1 = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_2} - \frac{\partial L}{\partial \phi_2} = 0 \qquad \Rightarrow$$

$$m_2(l - r_1)^2 \ddot{\phi}_2 \sin^2\theta_2 + 2m_2(r_1 - l)\dot{r}_1 \dot{\phi}_2 \sin^2\theta_2 + m_2(l - r_1)^2 \dot{\phi}_2 \dot{\theta}_2 \sin 2\theta_2 = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2} = 0 \qquad \Rightarrow$$

$$m_2(l - r_1)^2 \ddot{\theta}_2 - m_2(l - r_1)^2 \dot{\phi}_2^2 \dot{\theta}_2 \sin\theta_2 + m_2 g (l - r_1)\dot{\theta}_2 \sin\theta_2 = 0$$

ii) Erhaltungsgröße

Energie

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$$E = \dot{r}_1 \left[(m_1 + m_2)\dot{r}_1\right] + \dot{\phi}_1 \left[m_1 r_1^2 \dot{\phi}_1\right] + \dot{\phi}_2 \left[m_2 (l - r_1)^2 \dot{\phi}_2 \sin^2\theta_2\right]$$

$$+\dot{\theta}_2 \left[m_2 (l - r_1)^2 \dot{\theta}_2\right] - L$$

$$= \frac{m_1 + m_2}{2}\dot{r}_1^2 + \frac{m_1}{2} r_1^2 \dot{\phi}_1^2 + \frac{m_2}{2}(l - r_1)^2 \left[\dot{\phi}_2^2 \sin^2\theta_2 + \dot{\theta}_2^2\right] - m_2 g (l - r_1)\cos\theta_2$$

Drehimpulse

$$\ell_1 = \frac{\partial L}{\partial \dot{\phi}_1} = m_1 r_1^2 \dot{\phi}_1$$

$$\ell_2 = \frac{\partial L}{\partial \dot{\phi}_2} = m_2 (l - r_1)^2 \dot{\phi}_2 \sin^2\theta_2$$