Karlsruher Institut für Technologie

Institut für Theorie der Kondensierten Materie

Theorie der Kondensierten Materie II SS 2017

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Blatt 3

Lösungsvorschlag

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1. Fermionic chain (Kitaev model)

Consider spinless fermions on a one-dimensional chain of sites, numbered by an index n. The Hamiltonian reads $H = H_0 + V$, where

$$H_0 = \sum_n \left(t a_n^\dagger a_{n+1} + t a_{n+1}^\dagger a_n - \mu a_n^\dagger a_n \right)$$

and

$$V = \sum_n \left(\Delta a_n a_{n+1} + \Delta a_{n+1}^\dagger a_n^\dagger \right) \ .$$

Here t, Δ and μ are real constants.

(a) **Find the Green's function G_0 corresponding to H_0.** *Tip: use the Fourier representation.*

The Fourier transformation yields

$$H_0 = \int_{-\pi}^{\pi} \frac{dq}{2\pi} \epsilon_q a_q^\dagger a_q, \qquad V = i\Delta \int_{-\pi}^{\pi} \frac{dq}{2\pi} \sin q (a_{-q} a_q + a_{-q}^\dagger a_q^\dagger),$$

with

$$\epsilon_q = 2t \cos q - \mu.$$

The "non-interacting" problem is characterized by the "free" Green's function

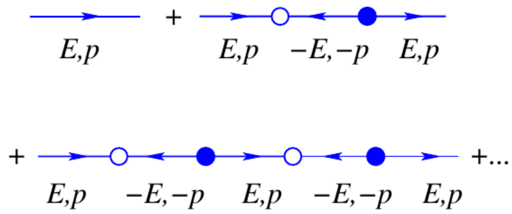
$$G_0(\epsilon, q) = \frac{1}{\epsilon - \epsilon_q + i\delta \mathrm{sign}\epsilon}.$$

(b) **Consider the perturbation series for the Green's function G of the full problem. Develop the diagrammatic rules. Sum up the series and determine the dispersion relation of the new excitations.**

The interaction potential V corresponds to two vertices in the diagram technique:

The expressions corresponding to the two vertices are $2i\Delta \sin q$ and $-2i\Delta \sin q$, respectively.

The perturbative corrections to the Green's function form the following series:



Notice, that the signs alternate. This follows form the momentum conservation. Because of this, only "even-order" corrections appear in the series. The elementary block is represented by the pair of vertices and a pair of Green's functions.

In terms of this block, the series is a simple geometric progression. This can be summed up using the standard rule. Thus we find

$$G(\epsilon, q) = \frac{G_0(\epsilon, q)}{1 + 4\Delta^2 \sin^2 q G_0(\epsilon, q) G_0(-\epsilon, -q)} = \frac{\epsilon + 2t\cos q - \mu}{\epsilon^2 - [(2t\cos q - \mu)^2 + 4\Delta^2 \sin^2 q] + i\delta}.$$

The poles of the full Green's function give the excitation spectrum:

$$\epsilon = \pm\sqrt{(2t\cos q - \mu)^2 + 4\Delta^2 \sin^2 q}.$$

(c) **Could the solution be found without perturbation theory?**

Since the Hamiltonian is quadratic, one can also use the Bogolyubov trnasformation known from the theory of superconductivity to solve the problem. This will be discussed in more detail in one of the next assignments.

(d) **The relation between the imaginary part of the Green's function and the occupation of the single-particle states**

The sign of the imaginary term $i\delta$ in $G(\epsilon, q)$ points out that the upper branch of the spectrum is empty, while the lower branch is fully occupied.

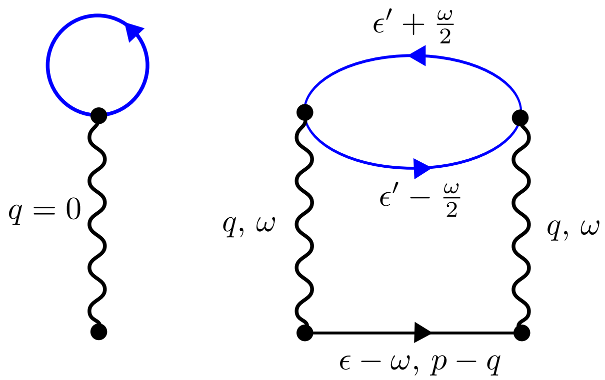
2. Heavy particle in the Fermi gas:

(a) **What is the maximal momentum transfer and the corresponding transferred energy ϵ_M?**

Suppose before the collision the light fermion has the momentum $\mathbf{p}_0$ and the heavy particle the momentum $\mathbf{p}_p$. The maximal momentum transfer appears when the collision is collinear. Considering the conservation laws of energy and momentum in the limit $M \gg m$ yields the momentum transfer $\Delta\mathbf{p}_p = 2\mathbf{p}_0$. The corresponding energy transfer is $\Delta\epsilon = 2(p_0^2 + \mathbf{p}_p\mathbf{p}_0)/M$. If we assume that the heavy particle is not too fast, $|\mathbf{p}_p| < p_0$, the typical energy transfer is of the order of $\epsilon_M \equiv 2p_0^2/M$. Since the momentum of the fermions is limited by the Fermi-momentum and $M \gg m$, the transferred energy is much smaller than the Fermi-energy.

(b) **Effective self-interaction of the heavy particle.**

Due to the contact-interaction with fermions the Green's function of the atom receives corrections. Those corrections can be incorporated in the usual way via the self-energy. The diagrams in the first and second order in λ are



Blue lines correspond to the fermions while the black line represents the free Green's function of the heavy particle. The correction to the Green's function is obtained by sandwiching the self-energy between the free Green's functions and summing the series. Thus the effect of the fermions is hidden in the self-energy that now describes an effective self-interaction of the atom. The first order correction is only a constant that can be skipped (no real interaction). The second order correction contains a fermion bubble (polarization operator). We calculate it in real space:

$$\Pi(\omega, \mathbf{r}) = -2i \int \frac{d\epsilon'}{2\pi} G(\epsilon' + \frac{\omega}{2}, \mathbf{r}) G(\epsilon' - \frac{\omega}{2}, \mathbf{r}) \tag{1}$$

The free fermion Green's function in real space reads

$$G(\epsilon, \mathbf{r}) = -\frac{m}{2\pi r} e^{i \mathrm{sign}(\epsilon) \kappa(\epsilon) r}, \qquad r = |\mathbf{r}|, \qquad \kappa(\epsilon) = \sqrt{p_0^2 + 2m\epsilon}. \tag{2}$$

Here p_0 is the Fermi momentum of the light fermions. As the above integral is divergent at large ϵ', we need to regularize it. The variable ω corresponds to the energy transfer of the collision. If $\omega = 0$ then there is no real collision. We thus say that

$$V_{eff}(\omega, \mathbf{r}) = \lambda^2 \left[\Pi(\omega, \mathbf{r}) - \Pi(0, \mathbf{r} \right] \tag{3}$$

describes a real interaction process. We can now split the integral into the regions $|\epsilon'| < p_0^2/2m$ and $|\epsilon'| > p_0^2/2m$. We aim at understanding the scattering at low transferred energy $\omega \ll E_F$. We can thus drop in the second region the dependence of the integrand on ω such that from this part there is no contribution. In the part of low energies we approximate $\kappa(\epsilon) \approx p_0$. Because of the sign-functions the integration region is limited to $-|\omega|/2 < \epsilon' < |\omega|/2$ (phase space). The result reads

$$V_{eff}(\omega, \mathbf{r}) = -\frac{\lambda^2 m^2 i |\omega|}{2\pi^3 r^2} \sin^2(p_0 r). \tag{4}$$

A Fourier transform to momentum space yields

$$V_{eff}(\omega, \mathbf{q}) = -\frac{2i\lambda^2 m^2 |\omega|}{\pi^2 q} \int_0^\infty \frac{dr}{r} \sin^2(p_0 r) \sin(qr) \tag{5}$$

$$= \frac{i\lambda^2 m^2 |\omega|}{2\pi^2 q} \int_0^\infty \frac{dr}{r} \left[\sin(2p_0 + q) r - \sin(2p_0 - q) r - 2 \sin qr \right] \tag{6}$$

If we now use

$$\int_0^\infty \frac{dx}{x}\sin(\alpha x) = \frac{\pi}{2}\text{sign}(\alpha), \tag{7}$$

we obtain

$$V_{eff}(\omega, \mathbf{q}) = -\frac{i\lambda^2 m^2|\omega|}{2\pi q}\theta(2p_0 - q) =: -i|\omega|F(q). \tag{8}$$

Here we find the result from (a) that the maximal transferred momentum in the collision is limited by $2p_0$.

(c) **Self-energy of the heavy particle.**
The self-energy is given by

$$\Sigma(\epsilon, \mathbf{p}) = i\int \frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3}G_0(\epsilon-\omega, \mathbf{p}-\mathbf{q})V_{eff}(\omega, \mathbf{q}). \tag{9}$$

Here G_0 is the free Green's function of the atom. Since we calculated V_{eff} only at small energies, the integration over ω does not converge. We need to introduce a factor that ensures convergence at large frequencies ω. We could for example replace the interaction by

$$V_{eff}(\omega, \mathbf{q}) = -i|\omega|\left(\frac{i\omega_0}{i\omega+|\omega|}\right)^n F(q) \tag{10}$$

where n is some large power that ensures that the integral converges and the main contribution comes from small ω; ω_0 is a cutoff frequency of the order of the Fermi energy. There is an additional subtlety related to the analytical properties of V_{eff}. V_{eff} should be analytic for $\omega > 0$ in the upper half plane while for $\omega < 0$ it should be analytic in the lower half plane. For $\omega < 0$ we will therefore subtract the analytic continuation of $\omega < 0$. We can now write the self energy in the following form

$$\Sigma(\epsilon, \mathbf{p}) = \int \frac{d^3q}{(2\pi)^3}A(\epsilon, \mathbf{p}-\mathbf{q})F(\mathbf{q}), \tag{11}$$

where

$$A(\epsilon, \mathbf{p}) = -\frac{1}{\pi}\int_0^{\omega_0} d\omega\frac{\omega}{T_{\epsilon,\mathbf{p}}+\omega} = \frac{T_{\epsilon,\mathbf{p}}}{\pi}\ln\left(\frac{\omega_0+T_{\epsilon,\mathbf{p}}}{T_{\epsilon,\mathbf{p}}}\right) - \frac{\omega_0}{\pi}, \tag{12}$$

and

$$T_{\epsilon,\mathbf{p}} = \frac{\mathbf{p}^2}{2M} - \epsilon - i0. \tag{13}$$

In the expression for A we can neglect the constant ω_0/π as well as T in the numerator of the argument of the logarithm. We further approximate

$$A(\epsilon, \mathbf{p}-\mathbf{q}) = \begin{cases} \frac{T_{\epsilon,\mathbf{p}}}{\pi}\ln\left(\frac{\omega_0}{T_{\epsilon,\mathbf{p}}}\right), & |\mathbf{q}| \ll |\mathbf{p}|, \\ \frac{1}{\pi}\left(T_{\epsilon,\mathbf{p}} + \frac{q^2}{2M} - \frac{\mathbf{qp}}{M}\right)\ln\left(\frac{2M\omega_0}{q^2}\right), & |\mathbf{q}| \gg |\mathbf{p}|. \end{cases} \tag{14}$$

In the case of large q we can drop the term $q^2/2M$ since it produces only a constant contribution to Σ as well the term $\mathbf{pq}/M$ since it vanishes when integrating over the directions of $\mathbf{q}$. We want to understand the behavior of Σ close to the pole $\epsilon \approx p^2/2M$ of the free Green's function G_0. This means that in the limit $\epsilon \ll \epsilon_M$ we are in the limit $p \gg 2p_0$. The integration over $|\mathbf{q}|$ is limited by $2p_0$ (see above) such that we need to use $A(\epsilon, \mathbf{p}-\mathbf{q})$ in the limit of small q. In the opposite limit

$\epsilon_M \ll \epsilon \ll \omega_0$ both limits contribute (split integral into two parts). We finally obtain for the self-energy

$$\Sigma(\epsilon, \mathbf{p}) = \alpha T_{\epsilon,\mathbf{p}} \begin{cases} \ln\left(\omega_0/\epsilon_M\right), & \epsilon \ll \epsilon_M, \\ \ln\left(\omega_0/T_{\epsilon,\mathbf{p}}\right), & \epsilon_M \ll \epsilon \ll \omega_0, \end{cases} \qquad \alpha = \frac{\lambda^2 m^2 p_0^2}{2\pi^4} \tag{15}$$

The Green's function is now obtained in the usual way

$$\tilde{G}_0(\epsilon, \mathbf{p}) = \frac{1}{\epsilon - \frac{p^2}{2M} - \Sigma(\epsilon, \mathbf{p})}. \tag{16}$$

Let us first discuss the case $\epsilon \ll \epsilon_M$. Here, the self-energy is proportional to $T_{\epsilon,\mathbf{p}}$ which means that the pole of the free Green's function is not changed. We only find a renormalization of the residuum Z of the Green's function:

$$\tilde{G}_0 = \frac{Z}{\epsilon - \frac{p^2}{2M} + i0}, \qquad Z = 1 + \alpha \ln(\omega_0/\epsilon_M) + \mathcal{O}(\alpha^2). \tag{17}$$

The case of large energies is more complicated since the logarithm is large close to the free pole $\epsilon \approx p^2/2M$:

$$\tilde{G}_0 = \frac{1 + \alpha \ln(\omega_0/T_{\epsilon,\mathbf{p}}) + \mathcal{O}(\alpha^2 \ln^2(\omega_0/T_{\epsilon,\mathbf{p}}))}{\epsilon - \frac{p^2}{2M} + i0}. \tag{18}$$

The simple perturbation theory breaks down and we should include also higher order diagrams in the self-energy [see part c)].

(d) **Renormalization of the Green's function in the regime $\epsilon_M \ll \epsilon \ll E_F$.**
In the limit of large energies we should include also higher order diagrams into the expansion of the self-energy. A convenient method to sum the most important contributions of the higher order diagrams (higher powers of logarithms) is the renormalization group. Similar to the last exercise sheet, we differentiate the Green's function with respect to the cutoff ω_0:

$$\frac{\partial \tilde{G}_0(\epsilon, \mathbf{p})}{\partial \omega_0} = \tilde{G}_0^2(\epsilon, \mathbf{p}) \frac{\partial \Sigma(\epsilon, \mathbf{p})}{\partial \omega_0} = \frac{\alpha}{\omega_0} \tilde{G}_0(\epsilon, \mathbf{p}). \tag{19}$$

The solution is of the form $\tilde{G}_0 \propto \omega_0^\alpha$. By considering the dimensions of the Green's function and the condition that for $\alpha = 0$ we should obtain the free Green's function we find

$$\tilde{G}_0(\epsilon, \mathbf{p}) = \frac{\omega_0^\alpha}{\left(\epsilon - \frac{p^2}{2M} + i0\right)^{\alpha+1}}. \tag{20}$$

Another way to qualitatively understand how the power-law emerges is to look at the higher order contributions. If there are some diagrams that produce higher powers of the logarithm and we include the correct combinatorics, we can sum those contributions and find a power-law:

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \ln^n(\omega_0/T_{\epsilon,\mathbf{p}}) = \exp\left\{\alpha \ln(\omega_0/T_{\epsilon,\mathbf{p}})\right\} = \left(\frac{\omega_0}{T_{\epsilon,\mathbf{p}}}\right)^\alpha. \tag{21}$$