Karlsruher Institut für Technologie | Institut für Theorie der Kondensierten Materie

# Übungen zur Theorie der Kondensierten Materie II  SS 18

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**Blatt 2**
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**1. Dyson equation** (25+25 Points)

We want to describe a particle on a lattice that interacts with an impurity at the origin. The sites of the lattice are numbered by an index $i$. Thus the state $|i\rangle$ refers to a position eigenstate located on site $i$.

(a) Consider a Hamiltonian $H$ that is the sum of two parts

$$H = H_0 + V = \sum_{ij} H_{ij}|i\rangle\langle j| + V_0|0\rangle\langle 0|.$$

Assume that we know everything about the problem involving the Hamiltonian $H_0$. We now want to study the effect of an impurity that sits at site 0. This is represented by the second term. The full single-particle Green's function is a matrix given by

$$G^{-1} = \omega - H_0 - V.$$

We define the bare Green's function $G_0$ as

$$G_0^{-1} = \omega - H_0.$$

Show that

$$G = \frac{1}{1 - G_0 V} G_0$$

holds. Now expand the fraction as a geometric series and compute the matrix element $\langle i|G|j\rangle$. You should find

$$\langle i|G|j\rangle = \langle i|G_0|j\rangle + V_0 \frac{\langle i|G_0|0\rangle\langle 0|G_0|j\rangle}{1 - V_0\langle 0|G_0|0\rangle}.$$

(25 pts)

(b) Insert for $\langle i|G_0|j\rangle$ the retarded free Green's function $\langle i|G_0^R|j\rangle$ in the continuum for a free particle with $\epsilon_k = k^2/(2m)$. It is

$$\langle x|G_0^R|x'\rangle = G_0^R(x - x') = \int \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{\omega - \epsilon_k + i\eta}\,. \tag{1}$$

Show that the full Green's function $\langle x|G^R|x'\rangle = G^R(x-x')$ is also a retarded Green's function, i.e. it has only poles at $\mathrm{Im}(\omega) < 0$. (25 Pts.)

**2. Aymptotic behavior of Green's functions** (25+25 Points)

In this problem we will analyze the large-$\omega$ behavior of the Green's functions $G^r_{AB}(\omega), G^a_{AB}(\omega)$ and $G^c_{AB}(\omega)$ for fermions $A = B^\dagger$.

(a) The Lehmann representation for $G^r(\omega)$ was derived in the lectures. Find a similar representation for $G^a(\omega)$ and $G^c(\omega)$ following the arguments used for $G^r(\omega)$. Schematically you will find representations of the form

$$\begin{aligned} G^r(\omega) &= \sum_{lm} \frac{N_r(l,m)}{\omega + E_l - E_m + i0^+} \\ G^a(\omega) &= \sum_{lm} \frac{N_a(l,m)}{\omega + E_l - E_m - i0^+} \\ G^c(\omega) &= \sum_{lm} \frac{N_1(l,m)}{\omega + E_l - E_m + i0^+} + \frac{N_2(l,m)}{\omega + E_l - E_m - i0^+}, \end{aligned}$$

for some appropriate $N_r, N_a, N_1$ and $N_2$ that you should determine. (25 Pts.)

(b) Now use the fermionic anti-commutation relations to show that

$$\sum_{l,m} N_r(l,m) = \sum_{l,m} N_a(l,m) = \sum_{l,m} [N_1(l,m) + N_2(l,m)] = 1$$

Use this to determine the asymptotic behavior of the Green's functions for $\omega \to \infty$. (25 Pts.)