Übungen zur Theorie der Kondensierten Materie II SS 18

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Blatt 3

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1. Orthogonality catastrophe II (40 Points)

On the last sheet we have seen that for the overlap of the wavefunctions of the perturbed and unperturbed Fermi gas holds

$$\ln |\langle\psi|\psi'\rangle|^2 \leq - \sum_{\epsilon_n \leq E_F, e_{m'} > E_F} |\langle n|m'\rangle|^2 .$$

Now, we want to estimate the behavior of the right hand side sum for large system sizes. For simplicity, let us assume that the gas is confined to a sphere of radius R and consider only the wave-functions with zero angular momentum $l = 0$. Show that the wave functions of the unperturbed (and non-interacting) Fermions are given by

$$\phi_n(r) = N_n \frac{\sin(k_n r)}{k_n r},$$

where r is the radial coordinate and $N_n = k/\sqrt{2\pi R}$. Which values is k_n allowed to take? Remeber quantum mechanical scattering theory: a perturbation potential will shift the phase of the wave functions. In the region near R (which is large for large R), the wavefunctions of the perturbed system can be approxiamted by

$$\phi'_n(r) = N_n \frac{\sin\left(k_n r + \delta_n\left(1 - \frac{r}{R}\right)\right)}{k_n r}.$$

Calculate $\langle n|m'\rangle$ approximately. Argue that

$$- \sum_{\epsilon_n \leq E_F, e_{m'} > E_F} |\langle n|m'\rangle|^2 \sim \ln N.$$

Hint: The sum over energies higher than the Fermi surface can be cut off at some index M, because the phase shifts δ_m become small as m increases. The energy scale E_M where this happens does obviously not depend on the system size but on the scattering properties of the perturbation. However, the momentum k_M does (via the radius $R \propto N^{1/3}$) depend on the system size, and we know that $k_M \sim M$.

2. Φ^4 - Theory (20 Points)

(a) We study the Φ^4-theory in $d = 0$ dimensions. The action is given by

$$S = S_0 + S_{int} = r\Phi^2 + \frac{u}{4}\Phi^4 .$$

As you have learned in the lecture, the expectation value of two Φ-fields is defined as

$$\langle \Phi\Phi \rangle = \frac{\int d\Phi\, \Phi^2 e^{-S}}{\int d\Phi\, e^{-S}}\,. \tag{1}$$

Determine this expectation value for small $\lambda = \frac{u}{r^2}$ using pertubation theory. Expand the expectation value up to the

- 2nd order in λ
- 10th order λ (computer aided).

(b) Evaluate the expection value Eq. (1) numerically as a function of λ, and compare the numerical result with the results of the perturbation theory obtained in (a). Plot the absolute value of the difference of the exact and perturbative results. Which conclusions on the validity of the perturbative approximation can you draw?

3. Gaussian integrals (40 Points)

(a) Show that for $Re\, a > 0$ it is

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}\,.$$

As the next step, demonstrate that

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2+bx} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}\,.$$

(b) Next we generalize the Gaussian integral to complex arguments. Show that

$$\int d(\bar{z},z) e^{-\bar{z}wz} = \frac{\pi}{w}, \qquad \mathrm{Re}\, w > 0,$$

where $\bar{z}$is the complex conjugate of z and $\int d(\bar{z},z) \equiv \int_{-\infty}^{\infty} dxdy$ is the independent integration over the real and imaginary parts $z = x + iy$. Next, proof the relation

$$\int d(\bar{z},z) e^{-\bar{z}wz+\bar{u}z+z\bar{u}} = \frac{\pi}{w} e^{\frac{\bar{u}v}{w}}, \qquad \mathrm{Re}\, w > 0.$$

(c) Use the above relations to verify the multi-dimensional generalization of the Gaussian intagral

$$\int d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}+\mathbf{j}^T\cdot\mathbf{v}} = (2\pi)^{N/2} \det \mathbf{A}^{-1/2} \exp[\frac{1}{2}\mathbf{j}^T\mathbf{A}\mathbf{j}],$$

where $\mathbf{A}$ is a positive definite, real, symmetric N-dimensional matrix, $\mathbf{v}$ is an N-component real vector and $\mathbf{j}$ is an arbitrary N-component vector. The last relation is used to derive Wick's theorem for real bosonic fields.