Theorie der Kondensierten Materie I WS 2012/2013

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1. Spin density wave instability (10 + 15 + 10 + 15 = 50 Punkte)

Consider the Hubbard model of electrons on the two-dimensional square lattice with Hamiltonian

$$H = -t \sum_{\langle i,j \rangle} \sum_{\sigma} \left(c^\dagger_{i\sigma} c_{j\sigma} + \text{h.c.} \right) + U \sum_i n_{i\uparrow} n_{i\downarrow} \,. \tag{1}$$

Here, $c^\dagger_{i\sigma}$ creates a fermion with spin $\sigma \in \{\uparrow, \downarrow\}$ at lattice site $i = (i_1, i_2)$ with $i_{1,2} \in \mathbb{N}$, and $n_{i\sigma} = c^\dagger_{i\sigma} c_{i\sigma}$. The sum over $\langle i, j \rangle$ sums once over nearest-neighbors on the 2D square lattice with Bravais lattice vectors $\boldsymbol{a}_1 = a(1,0)$ and $\boldsymbol{a}_2 = a(0,1)$. Take the hopping amplitude $t > 0$ and the on-site repulsion $U \geq 0$.

(a) Set $U = 0$ and calculate the bandstructure (see homework 4) and the density of states $\rho(\epsilon)$. The density of states exhibits a van-Hove singularity at $\epsilon = 0$ (at a half-filled band). Find the divergence to logarithmic accuracy (similar to homework 6).

(b) Now consider $U > 0$ and use the Hartree-Fock decoupling to obtain a quadratic mean-field Hamiltonian H_{HF}. Use the mean-field order parameter by pairing (anti-ferromagnetic spin density wave channel)

$$\left\langle n_{(i_1,i_2),\uparrow} \right\rangle = n + (-1)^{i_1+i_2} m; \quad \left\langle n_{(i_1,i_2),\downarrow} \right\rangle = n - (-1)^{i_1+i_2} m \,. \tag{2}$$

(c) In the last two parts of the exercise, we consider the system at half-filling, *i.e.*, the average number of electrons per site is one particle per site. Calculate the energies of the resulting mean-field Hamiltonian H_{HF}.

(d) Now consider the system at zero temperature $T = 0$. Determine the order parameter m (spin density wave amplitude) by minimizing the free energy:

$$\frac{\partial}{\partial m} \langle H_{HF} - \mu N \rangle_{T=0} = 0 \,, \tag{3}$$

where $N = \sum_{j,\sigma} n_{j,\sigma}$ is the total number of particles in the system and μ is the chemical potential that controls N. During the calculation use that $\mu = Un$ to ensure half-filling. Solve the resulting gap equation to logarithmic accuracy, *i.e.*, take the logarithmic expression for the density of states and solve the integral only to logarithmic accuracy (extract the logarithmic divergence and neglect any numerical prefactors in front of the logarithm).

2. Spin susceptibility in a Fermi liquid. (10 + 20 + 20 = 50 Punkte)

A system of spin-1/2 fermions, with zero charge, quadratic spectrum $\varepsilon_{\mathbf{k}} = \frac{k^2}{2m}$, and weak contact interaction $V(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')$ between the particles, is placed in magnetic field B at $T = 0$. Because the particles are not charged, the field does not affect their orbital motion and acts on the spins only,

$$\hat{\mathcal{H}}_B = \mu B \sum_{\mathbf{k}} \left(\hat{a}^\dagger_{\mathbf{k}\uparrow} \hat{a}_{\mathbf{k}\uparrow} - \hat{a}^\dagger_{\mathbf{k}\downarrow} \hat{a}_{\mathbf{k}\downarrow} \right).$$

The Fermi energy is large, $E_F \gg \mu B$.

(a) Write the Hamiltonian of the interaction between the particles in the second quantization representation.

(b) Show that the interaction leads to an effective shift of the energies of the spin-down (-up) fermions, $\delta E_\downarrow = \gamma g n_\uparrow$ ($\delta E_\uparrow = \gamma g n_\downarrow$), where $n_\uparrow$ ($n_\downarrow$) is the concentration of the fermions with spin up (down). Find constant γ.

(c) Evaluate the magnetic susceptibility of the system.