KARLSRUHER INSTITUT FÜR TECHNOLOGIE

INSTITUT FÜR THEORIE
DER KONDENSIERTEN MATERIE

Theorie der Kondensierten Materie I   WS 2012/2013

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1. Phonons on the triangular lattice (40 Punkte)

Calculate the phonon spectrum for the two-dimensional triangular lattice, which is a Bravais lattice spanned by the unit vectors $\boldsymbol{a}_1 = \frac{a}{2}(1, \sqrt{3})$ and $\boldsymbol{a}_2 = \frac{a}{2}(-1, \sqrt{3})$. Assume that atoms sit at the sites of the triangular lattice, have mass m and can only move within the 2D plane. Assume that only nearest-neighbor atoms interact and use the harmonic approximation for the resulting potential (with spring constant k).

2. The two-site Hubbard model $(15 + 15 + 15 + 15 = 60$ Punkte)

This problem concerns the two-site Hubbard model, which you will solve exactly. Its Hamiltonian is

$$H = -2t \sum_{\sigma} \left(c^{\dagger}_{1\sigma} c_{2\sigma} + c^{\dagger}_{2\sigma} c_{1\sigma} \right) + U \sum_{j=1}^{2} c^{\dagger}_{j\uparrow} c_{j\uparrow} c^{\dagger}_{j\downarrow} c_{j\downarrow} \, . \tag{1}$$

Here t is the hopping amplitude between nearest-neighbor sites, $c^{\dagger}_{j\sigma}$ creates a particle of spin $\sigma \in \{\uparrow, \downarrow\}$ on lattice site j and $U > 0$ denotes the on-site repulsion. We take periodic boundary coefficients which is responsible for the factor of 2 in the first term.

(a) Construct the full basis of states for the model using the position-space creation operators,

$$|0\rangle, c^{\dagger}_{1\uparrow}|0\rangle, c^{\dagger}_{1\uparrow} c^{\dagger}_{2\downarrow}|0\rangle, \text{etc}\, . \tag{2}$$

How many are there ? (*Hint*: make sure that all states in the set you write down are linearly independent.)

(b) Show that the total number of electrons

$$N = \sum_{j\sigma} c^{\dagger}_{j\sigma} c_{j\sigma} \tag{3}$$

and that the total z-projection of the spin

$$S^z = \frac{1}{2} \sum_{j} \left(c^{\dagger}_{j\uparrow} c_{j\uparrow} - c^{\dagger}_{j\downarrow} c_{j\downarrow} \right) \tag{4}$$

are both conserved quantities of the Hamiltonian (1). Hence divide the space of states spanned by the basis (2) into subspaces of fixed N and S^z.

(c) Diagonalize H within each of these subspaces, and hence obtain all energy eigenvalues and eigenstates of the Hamiltonian.

(d) Analyze your solution in the particular cases: (i) $U \ll t$ and (ii) $U \gg t$. Pay particular attention to the $N = 2$ subspace.