Microscopic Theory of Superconductivity WS 2014/15

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Sheet 03
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1. Fermionic oscillator $(10 + 10 + 10 + 10 = 40$ points)

Let Ψ and $\Psi^\dagger$ be two fermionic operators obeying anticommutation relations $\{\Psi, \Psi^\dagger\} = \Psi\Psi^\dagger + \Psi^\dagger\Psi = 1$.

(a) Define the number operator $N = \Psi^\dagger\Psi$ and show that $N^2 = N$. Which eigenvalues does N have?

(b) Define the two number eigentstates $N|0\rangle = 0|0\rangle$ and $N|1\rangle = 1|1\rangle$. Show that $\Psi^\dagger|0\rangle = 1$ and $\Psi|1\rangle = |0\rangle$.

(c) Consider a Fermi oscillator with Hamiltonian $H_0 = \omega\Psi^\dagger\Psi$. Explicitly compute the partition function $Z = \text{Tr}e^{-\beta(H_0-\mu N)} = e^{-\beta F(\mu,\beta)}$ and the average particle number $N = -\frac{\partial F}{\partial \mu}$.

(d) Consider a toy Hubbard model

$$H_0 = \omega(\Psi_1^\dagger\Psi_1 + \Psi_2^\dagger\Psi_2) + U(\Psi_1^\dagger\Psi_1\Psi_2^\dagger\Psi_2)\,. \tag{1}$$

Compute the partition sum Z and the average particle number N.

2. Fermion coherent states and Grassmann numbers $(5+5+10+10+10+10+10 = 60$ points)

The rules for manipulating Grassmann numbers ψ, ψ' are: (i) all Grassmann numbers anticommute with each other and with all fermionic operators; (ii) when a Grassmann number is taken through a ket or brai containing an odd (even) number of fermions it will (not) change sign; (iii) integrals over Grassmann numbers follow from the two definitions $\int \psi d\psi = 1$ and $\int 1 d\psi = 0$. Note that $d\psi$ is also a Grassmann number. One should not associate a numerical value with Grassmann numbers, all you need are the definitions above.

(a) Consider a state $|\psi\rangle = |0\rangle - \psi|1\rangle$, where the states $|0\rangle, |1\rangle$ were defined above. Show that this state is an eigenstate of the annihilation operator with eigenvalue ψ: $\Psi|\psi\rangle = \psi|\psi\rangle$. The state $|\psi\rangle$ is thus called a fermionic coherent state (in analogy to bosonic coherent states which are eigenstates of a bosonic annihilation operator).

(b) Consider the state $\langle\bar\psi| = \langle 0| - \langle 1|\bar\psi$. Show that this is a left eigenstate of the creation operator with eigenvalue $\bar\psi$, *i.e.*, $\langle\bar\psi|\Psi^\dagger = \langle\bar\psi|\bar\psi$.

(c) Compute the inner product $\langle\bar\psi|\psi\rangle$ and reexponentiate your result.

(d) Calculate the Gaussian integrals over Grassmann numbers $\int e^{-a\bar\psi\psi} d\bar\psi d\psi$, and the generalization $\int e^{-\bar\psi M\psi} d\bar\psi_1 d\psi_1 d\bar\psi_2 d\psi_2$ to a two-component Grassmann vector $\psi = (\psi_1, \psi_2)$, $\bar\psi = (\bar\psi_1, \bar\psi_2)$ with 2×2 matrix M.

(e) Calculate the expectation values

$$\langle \bar{\psi}_i \psi_j \rangle = \frac{\int \bar{\psi}_i \psi_j e^{a_1 \bar{\psi}_1 \psi_1 + a_2 \bar{\psi}_2 \psi_2} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}{\int e^{a_1 \bar{\psi}_1 \psi_1 + a_2 \bar{\psi}_2 \psi_2} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}$$

$$\langle \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l \rangle = \frac{\int \bar{\psi}_i \bar{\psi}_j \psi_k \psi_l e^{a_1 \bar{\psi}_1 \psi_1 + a_2 \bar{\psi}_2 \psi_2} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2}{\int e^{a_1 \bar{\psi}_1 \psi_1 + a_2 \bar{\psi}_2 \psi_2} d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2} \,.$$

(f) Show the following resolution of identity

$$1 = \int |\psi\rangle \langle \bar{\psi}| e^{-\bar{\psi}\psi} d\bar{\psi} d\psi \,.$$

(g) Evaluate the partition function for the fermionic oscillator

$$Z = \mathrm{Tr} e^{-\beta(\omega-\mu)\Psi^\dagger \Psi} = \int \langle -\bar{\psi}| e^{-\beta(\omega-\mu)\Psi^\dagger \Psi} |\psi\rangle e^{-\bar{\psi}\psi} d\bar{\psi} d\psi \,,$$

and compare with your result from the previous question.