Lecture Notes: Field Theory of Disordered Systems

Jörg Schmalian

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Institute for Theory of Condensed Matter, Karlsruhe Institute of Technology

Part I

Quenched disorder and the replica method

Real solids are affected by imperfections of the periodic arrangement of atoms. Vacancies and accidentally or deliberately induced disorder strongly affect the electronic, magnetic etc. properties of a given material. In most cases the system is unable to "heal" the randomness due to disorder, as the characteristic time scales for the diffusion of an ill-placed atom tend to be much larger than many of the internal degrees of freedom. A system is then characterized by a given, quenched (i.e. fixed) disorder configuration that undergoes no dynamics and is therefore frozen-in-time. The Hamiltonian for such a system is given as

$$H\left[\sigma, w\right] = H_0\left[\sigma\right] + W\left[\sigma, w\right], \tag{1}$$

where σ stands for the microscopic degrees of freedom that equilibrate on the time scale of a typical measurement, while w stands for the disorder configuration. For example, in the case a disordered electron gas σ stands for the positions $\mathbf{x}_i$ and spins α_i of the electrons and

$$W = \sum_i w\left(\mathbf{x}_i\right) = \int d^d x \sum_\alpha w\left(\mathbf{x}\right) \psi_\alpha^\dagger\left(\mathbf{x}\right) \psi_\alpha\left(\mathbf{x}\right) \tag{2}$$

is characterized by a random single-particle potential $w\left(\mathbf{x}\right)$.

The probability distribution of the disorder is $P\left[w\right]$ with measure Dw such that

$$\int Dw P\left[w\right] = 1. \tag{3}$$

Usually, we consider uncorrelated disorder of the type

$$\overline{w\left(\mathbf{x}\right)w\left(\mathbf{x}'\right)}=g\delta\left(\mathbf{x}-\mathbf{x}'\right),\tag{4}$$

with Gaussian distribution function

$$P\left[w\right]\propto\exp\left(-\frac{1}{2g}\int d^dxw\left(\mathbf{x}\right)^2\right).\tag{5}$$

Consider now the quantum mechanical expectation value X_w of some extensive macroscopic variable X. X_w depends on the given disorder configuration w. For most practical applications is it sufficient to consider disorder averages such as the mean value

$$\overline{X_w}=\int DwP\left[w\right]X_w.\tag{6}$$

This is in particular the case for self-averaging variables X which obey:

$$\lim_{\mathcal{N}\to\infty}R_X\to0.\tag{7}$$

Here we use the relative variance

$$R_X\equiv\frac{\overline{X_w^2}-\overline{X_w}^2}{\overline{X_w}^2},\tag{8}$$

while $\mathcal{N}$ is the system size. One justification for performing the average is that one may subdivide a macroscopic system in a sufficiently large set of smaller, but still macroscopic subsystems. In each subsystem the disorder realization is statistically similar but individually distinct. Averaging over these subsystems should then yield information about the behavior of the entire system. In case of a macroscopic variable with short range interactions and correlations, the law of large numbers immediately implies that

$$R_X\propto\mathcal{N}^{-1},\tag{9}$$

leading to what is referred to as strong self-averaging. As we will see, near a critical point the decay of R_X with system size can be severely slowed down and in some cases it even follows that $\lim_{\mathcal{N}\to\infty}R_X=const.$ with non zero constant. This corresponds to a break-down of self-averaging. Even in this case of no self-averaging, where an entire distribution function of observables becomes crucial, it is still sensible to analyze mean values of observables.

Quantities that should be averaged are macroscopic extensive observables, such as the energy, the heat capacity, the susceptibility or correlation functions. Thus, we should for example average the free energy:

$$F_{av.}=\overline{F_w}=-T\overline{\log Z_w},\tag{10}$$

where

$$Z_w=\mathrm{tr}_\sigma e^{-\beta H[\sigma,w]}\tag{11}$$

is the partition sum of a given disorder realization. This is qualitatively different from an average of the partition function $\overline{Z_w}$ and then taking the logarithm. Note, the average $\overline{Z_w}$ can be performed rather naturally by using the usual field-theoretical methods of many-body theory. One calls the average $\overline{Z_w}$ an annealed average while $\overline{\log Z_w}$ corresponds to a quenched average.

In order to perform the average over the logarithm of the partition function we use the replica method. At the heart of the method is the identity

$$\log Z_w = \lim_{n \to 0} \frac{1}{n} \left(Z_w^n - 1 \right), \tag{12}$$

which expresses the logarithm as a limit of small power. The logic is to evaluate averages of Z_w^n for arbitrary integer n and then analytically continue the result to $n \to 0$. Thus, we first analyze for integer n:

$$Z_w^n = \left(\mathrm{tr}_\sigma e^{-\beta H[\sigma, w]} \right)^n = \mathrm{tr}_{\{\sigma_i\}} e^{-\beta \sum_{i=1}^n H[\sigma_i, w]}. \tag{13}$$

The average

$$\overline{Z_W^n} = \overline{\mathrm{tr}_{\{\sigma_i\}} e^{-\beta \sum_{i=1}^n H[\sigma_i, W]}} \tag{14}$$

looks like an annealed average of an extended (replicated) system. The replica index i enters the theory like an additional flavor index. Now we define the corresponding replicated free energy.

$$F_n \equiv -\frac{T}{n} \log \overline{Z_w^n}. \tag{15}$$

It holds

$$
\begin{aligned}
\lim_{n \to 0} F_n &= -T \lim_{n \to 0} \frac{1}{n} \log \overline{Z_w^n} \\
&= -T \lim_{n \to 0} \frac{1}{n} \log \overline{\exp\left(n \log Z_w \right)} \\
&= -T \lim_{n \to 0} \frac{1}{n} \log \overline{\left(1 + n \log Z_w \right)} \\
&= -T \overline{\log Z_w} = F_{av}
\end{aligned}
\tag{16}
$$

Thus, we managed to express an average of the logarithm of the partition function in terms of an average of the partition function, yet with the additional flavor (replica) index for all observables.

While the replica method appears rather technical, there is a simple physical motivation for the procedure. Consider first the free energy of a given disorder configuration

$$F_w = -T \log Z_w. \tag{17}$$

Suppose now that the disorder is in equilibrium as well, yet at a different temperature T'. Then we would obtain a partition function

$$Z' = \int Dw P[w] e^{-\beta' F_w} = \overline{Z_w^n} \tag{18}$$

where now

$$n = \frac{T}{T'}.$$ (19)

The corresponding free energy is

$$
\begin{aligned}
F' &= -T' \log Z' \\
&= -\frac{T}{n} \log \overline{Z_w^n},
\end{aligned}
$$ (20)

i.e. the same expression that enters the replica formalism in Eq.16. We realize that the quenched average of the free energy corresponds to the limit $T' \to \infty$, where the internal degrees of freedom have no feedback on and cause no correlation of the disorder distribution. The disorder is so uncorrelated that it appears to originate from a much higher temperature and was then frozen in. This is the reason why we call this average quenched, as it originates from a much higher temperature as had no time to truly equilibrate. The quenched average is therefore nothing but

$$F_{av} = \lim_{T' \to \infty} F',$$ (21)

which offers a rather simple physical interpretation for the replica formalism.

Part II
Summary of critical behavior in clean systems

1 Classical critical systems

The continuum's version of the Ising model is the classical ϕ^4-theory determined by the effective Hamiltonian

$$H[\phi] = \frac{1}{2} \int d^d x \left(r_0 \phi^2 + (\nabla \phi)^2 + \frac{u}{2} \phi^4 \right).$$ (22)

It determines the partition function

$$Z = \int D\phi \exp\left(-\beta H[\phi]\right).$$ (23)

Here $D\phi = \prod_k d\phi_k$ where ϕ_k are the Fourier modes of the field $\phi(x)$. If the underlying microscopic theory is defined on a lattice with lattice constant a, no Fourier coefficients with $k > 2\pi/a$ should occur. In addition, the above effective

Hamiltonian should be understood as a coarse grained version of the theory, i.e. it should be valid only for wavelengths that are longer that the typical microscopic length scale. This gives rise to an upper momentum cut off Λ that is smaller that $2\pi/a$. Modes are suppressed for $k > \Lambda$.

There is no reason to confine ourselves to ϕ being a real number. For example, in case of continuously varying spins, as described by the XY- or Heisenberg models, the order parameter is an N-component vector $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$, with $N = 2$ or 3, respectively. It is often interesting and convenient to consider N-component order parameters with arbitrary N. Then the ϕ^4-model becomes

$$H[\phi] = \frac{1}{2} \int d^d x \left(r_0 \phi \cdot \phi + \nabla \phi \cdot \nabla \phi + \frac{u}{2} (\phi \cdot \phi)^2 \right) \tag{24}$$

where $\phi \cdot \phi = \sum_{i=1}^{N} \phi_i^2$.

1.1 Lower critical dimension

The role of fluctuations depends sensitively on the dimension d. As we will see, the space-dimension where fluctuations become important is referred to as the upper critical dimension d_{uc}. In case of the ϕ^4-theory we will find that $d_{uc} = 4$. The notion of an upper critical dimension makes only sense if there is in fact long range order and a finite transition temperature. For example, in case of the one-dimensional Ising model we found that no finite transition temperature exists. Whether long range order occurs or not can we estimated rather easily and determines the lower critical dimension d_{lc}. We will now show that $d_{lc} = 1$ for systems with discrete order parameter symmetry (and short range interactions), while $d_{lc} = 2$ for systems with continuous order parameter.

Let us first consider a discrete order parameter, like the Ising model. Suppose we have an ordered state with a given order parameter $\phi(x) = \phi_0$. Now lets estimate the free energy cost of a defect of the perfectly ordered state. In case of the Ising model (i.e. scalar ϕ^4-model) such a defect could be a droplet of size L. Suppose inside the droplet holds $\phi(x) = -\phi_0$, while outside of it the order parameter has the opposite sign. A typical configuration would be

$$\phi(x) \simeq \phi_0 \tanh\left(\frac{x - L}{l_0} \right) \tag{25}$$

where l_0 is some microscopic length scale. It follows for the energy of the defect

$$E_{defect} = \int d^d x \, (\nabla \phi)^2 \simeq \phi_0^2 \frac{1}{l_0} L^{d-1} \tag{26}$$

The typical energy of such a defect is

$$F_{defect} \simeq \phi_0^2 \frac{1}{l_0} L^{d-1} - T \log L^d, \tag{27}$$

where the second term is the entropy associated with the possible arrangement of the droplet. Such droplets will certainly occur and they are the natural

mechanism that reduce then order parameter to a smaller but finite value at $0 < T < T_c$. As long as $d > 1$, the interface-energy will always dominate over the entropy-gain of the defect, making large droplets increasingly unlikely. However, for $d = 1$ holds that the interface energy is independent of size (the surface consists of only two points no matter how large the droplet size L). There is nothing to prevent arbitrarily large droplets and droplets in other droplets etc. to occur. Thus, long range order will not occur. This is of course in complete agreement with our exact result for the one-dimensional Ising model. It holds $d_{\mathrm{lc}} = 1$ as mentioned above.

Next we consider a continuously varying order parameter, i.e. $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$. In comparison to the scalar order parameter, we can now simply rotate the order parameter vector without changing its amplitude. For example

$$\phi = \phi_0 \left(\cos \theta \left(x \right), \sin \theta \left(x \right), \cdots, 0 \right) \tag{28}$$

with

$$\theta \left(x \right) = \begin{cases} \pi \left(1 - \frac{|x|}{L} \right) & |x| < L \\ 0 & |x| \geq L \end{cases} \tag{29}$$

Then, the order parameter outside the defect is $\phi \left(|x| \geq L \right) = (1, 0, \cdots, 0)$ and it slowly rotates to reach $\phi \left(x = 0 \right) = (0, 1, \cdots, 0)$. Since $\phi \cdot \phi = \phi_0^2$ the nonlinear ϕ^4 term is completely unaffected by this defect, in distinction to the single component order parameter that had to "climb over the hill" of the energy landscape and locally melt to zero. The penalty comes from

$$E_{defect} = \int d^d x \left(\nabla \phi \right)^2 = \phi_0^2 \int d^d x \left(\nabla \theta \right)^2 \simeq \phi_0^2 L^{d-2}, \tag{30}$$

yielding

$$F_{defect} \simeq \phi_0^2 L^{d-2} - T \log L^d. \tag{31}$$

The same reasoning as before leads to $d_{\mathrm{lc}} = 2$ for systems with continuously varying order parameter. No long range order is possible in $d = 2$ and below. This statement can be proven rigorously and goes back to Hohenberg as well as Mermin and Wagner.

1.2 Landau theory of phase transitions

A first attempt to solve these problems is to approximate the integral by the dominant contribution of the integrand, i.e. we write

$$\int D\phi \exp \left(-\beta H \left[\phi \right] \right) \simeq \exp \left(-\beta H \left[\phi_0 \right] \right) \tag{32}$$

where $\left. \frac{\delta H}{\delta \phi} \right|_{\phi = \phi_0} = 0$. This leads to the Landau theory of phase transitions. Of course, in general, it is not only the minimum of $H \left[\phi \right]$ w.r.t. ϕ which corresponds to physically realized configurations. Instead one has to integrate over all values

of ϕ to obtain the free energy. Before we do this, we briefly discuss the Landau theory.

Landau proposed that one should introduce an order parameter to describe the properties close to a phase transition. This order parameter should vanish in the high temperature phase and be finite in the ordered low temperature phase. The mathematical structure of the order parameter depends strongly on the system under consideration. In case of an Ising model the order parameter is a scalar, in case of the Heisenberg model it is a vector. For example, in case of a superconductor or the normal fluid - superfluid transition of ^{4}He it is a complex scalar, characterizing the wave function of the coherent low temperature state. Another example are nematic liquid crystals where the order parameter is a second rank tensor.

In what follows we will first develop a Landau theory for a scalar, Ising type order. Landau argued that one can expand the free energy density in a Taylor series with respect to the order parameter ϕ. This should be true close to a second order transition where ϕ vanishes continuously:

$$\mathcal{H}(\phi) = -h\phi + \frac{r_0}{2}\phi^2 + \frac{b}{3}\phi^3 + \frac{u}{4}\phi^4 + ... \qquad (33)$$

The physical value of the order parameter is the determined by minimizing $\mathcal{H}(\phi)$

$$\left. \frac{\partial \mathcal{H}(\phi)}{\partial \phi} \right|_{\phi=\phi_0} = 0. \qquad (34)$$

If $u < 0$ this minimum will be at $\pm\infty$ which is unphysical. If indeed $u < 0$ one needs to take a term $\sim \phi^6$ into account and see what happens. In what follows we will always assume $u > 0$. In the absence of an external field should hold that $\mathcal{H}(\phi) = \mathcal{H}(-\phi)$, implying $h = b = 0$. Whether or not there is a minimum for $\phi \neq 0$ depends now on the sign of r_0. If $r_0 > 0$ the only minimum of

$$\mathcal{H}(\phi) = f_0 + \frac{r_0}{2}\phi + \frac{u}{4}\phi^4 \qquad (35)$$

is at $\phi = 0$. However, for $r_0 < 0$ there are two a new solutions $\phi = \pm\sqrt{\frac{-r_0}{u}}$. Since ϕ is expected to vanish at $T = T_c$ we conclude that $r_0(T)$ changes sign at T_c suggesting the simple ansatz

$$r_0(T) = a_0(T - T_c) \qquad (36)$$

with $a_0 > 0$ being at most weakly temperature dependent. This leads to a temperature dependence of the order parameter

$$\phi_0 = \begin{cases} \sqrt{\frac{a_0(T_c-T)}{u}} & T < T_c \\ 0 & T > T_c \end{cases} . \qquad (37)$$

It will turn out that a powerlaw relation like

$$\phi \sim (T_c - T)^\beta \qquad (38)$$

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is valid in a much more general context. The main change is the value of β. The prediction of the Landau theory is $\beta = \frac{1}{2}$.

Next we want to study the effect of an external field (= magnetic field in case ϕ characterizes the magnetization of an Ising ferromagnet). This is done by keeping the term $h\phi$ in the expansion for f. The actual external field will be proportional to h. Then we find that f is minimized by

$$r_0 \phi_0 + u\phi_0^3 = h \tag{39}$$

Right at the transition temperature where $r_0 = 0$ this gives

$$\phi_0 \sim h^{1/\delta} \tag{40}$$

where the Landau theory predicts $\delta = 3$. Finally we can analyze the change of the order parameter with respect to an external field. We introduce the susceptibility

$$\chi = \left. \frac{\partial \phi_0}{\partial h} \right|_{h \to 0} \tag{41}$$

and find from Eq.39

$$r_0 \chi + 3u\phi_0^2 \left(h = 0 \right) \chi = 1 \tag{42}$$

using the above result for $\phi_0^2 \left(h = 0 \right) = \frac{r_0}{u}$ if $T < T_c$ and $\phi_0^2 \left(h = 0 \right) = 0$ above T_c gives

$$\chi = \begin{cases} \frac{1}{4a_0} \left(T_c - T \right)^{-\gamma} & T < T_c \\ \frac{1}{a_0} \left(T - T_c \right)^{-\gamma} & T > T_c \end{cases} \tag{43}$$

with exponent $\gamma = 1$.

Next we consider the specific heat where we insert our solution for ϕ_0 into the free energy density.

$$\mathcal{H} \left(\phi_0 \right) = \frac{r_0}{2} \phi_0^2 + \frac{u}{4} \phi_0^4 = \begin{cases} -\frac{a_0^2}{4u} \left(T - T_c \right)^2 & T < T_c \\ 0 & T > T_c \end{cases} \tag{44}$$

This yields for the specific heat per volume

$$C = -T \frac{\partial^2 f}{\partial T} = -T \frac{\partial^2 \mathcal{H} \left(\phi_0 \right)}{\partial T} = \begin{cases} \frac{a_0^2}{4u} T & T < T_c \\ 0 & T > T_c \end{cases} . \tag{45}$$

The specific heat is discontinuous. As we will see later, the general form of the specific heat close to a second order phase transition is

$$C \left(T \right) \sim \left(T - T_c \right)^{-\alpha} + const \tag{46}$$

where the result of the Landau theory is

$$\alpha = 0. \tag{47}$$

In our analysis of the Landau theory we only considered spatially homogeneous solutions of the order parameter. Next, we include the more general case of spatially varying order parameters, i.e.

$$H = \int d^d x \mathcal{H}[\phi] \tag{48}$$

is given as

$$\mathcal{H}[\phi] = \frac{r_0}{2}\phi(x)^2 + \frac{u}{4}\phi(x)^4 - h(x)\phi(x) + \frac{1}{2}(\nabla\phi(x))^2 \tag{49}$$

where we assumed that it costs energy to induce an inhomogeneity of the order parameter. In addition we assumed that we can always absorb the coefficient $\frac{1}{2}(\nabla\phi(x))^2$ into the definition of the order parameter. The variational minimum $\left.\frac{\delta H}{\delta\phi}\right|_{\phi=\phi_0} = 0$ of H is now determined by the Euler-Lagrange equation

$$\frac{\partial\mathcal{H}}{\partial\phi} - \nabla\frac{\partial\mathcal{H}}{\partial\nabla\phi} = 0 \tag{50}$$

which leads to the nonlinear partial differential equation

$$r_0\phi(x) + u\phi(x)^3 = h(x) + \nabla^2\phi(x). \tag{51}$$

Above the transition temperature we neglect again the non-linear term and have to solve

$$r_0\phi(x) - \nabla^2\phi(x) = h(x) \tag{52}$$

It is useful to consider the generalized susceptibility

$$\delta\phi(x) = \int d^d x'\chi(x-x')\,\delta h(x') \tag{53}$$

which determines how much a local change in the order parameter is affected by a local change of an external field at a distance $x - x'$. This is often written as

$$\chi(x-x') = \frac{\delta\phi(x)}{\delta h(x')}. \tag{54}$$

We determine $\chi(x-x')$ by Fourier transforming the above differential equation with

$$\phi(x) = \int \frac{d^d k}{(2\pi)^d}e^{-ikx}\phi(k) \tag{55}$$

which gives

$$r_0\phi(k) + k^2\phi(k) = h(k) \tag{56}$$

In addition it holds for $\chi(k)$:

$$\delta\phi(k) = \chi(k)\,\delta h(k). \tag{57}$$

This leads to

$$\chi(k) = \frac{1}{\xi^{-2} + k^2} \tag{58}$$

where we introduced the length scale

$$\xi = \sqrt{\frac{1}{r_0}} = \sqrt{\frac{1}{a_0}}(T - T_c)^{-1/2} \tag{59}$$

This result can now be back-transformed yielding at large distances

$$\chi(x - x') = \left(\frac{\xi}{|x - x'|}\right)^{\frac{d-1}{2}} \exp\left(-\frac{|x - x'|}{\xi}\right) \tag{60}$$

Thus, spins are not correlated anymore beyond the correlation length ξ. In general the behavior of ξ close to T_c can be written as

$$\xi \sim (T - T_c)^{-\nu} \tag{61}$$

with $\nu = \frac{1}{2}$. A similar analysis can be performed in the ordered state. Starting again at

$$r_0\phi(x) + u\phi(x)^3 = h(x) + \nabla^2\phi(x) \tag{62}$$

and assuming $\phi(x) = \phi_0 + \psi(x)$ where ϕ_0 is the homogeneous, $h = 0$, solution, it follows for small $\psi(r)$:

$$\left(r_0 + 3u\phi_0^2\right)\psi(x) = h(x) + \nabla^2\psi(x) \tag{63}$$

and it holds $r_0 + 3u\phi_0^2 = -2r_0 > 0$. Thus in momentum space

$$\chi(k) = \frac{d\psi(k)}{dh(k)} = \frac{1}{\xi_<^{-2} + k^2} \tag{64}$$

with

$$\xi = \sqrt{\frac{1}{-2r_0}} = \sqrt{\frac{1}{2a_0}}(T_c - T)^{-1/2} \tag{65}$$

1.3 Ginzburg criterion

One can now estimate the range of applicability of the Landau theory. This is best done by considering the next order corrections and analyze when they are small. If this is the case, one can be confident that the theory is controlled. Before we go into this we need to be able to perform some simple calculations with these multidimensional integrals.

First we consider for simplicity a case where $H_{\text{eff}}[\phi]$ has only quadratic contributions. It holds

$$Z = \int D\phi \exp\left(-\frac{1}{2}\sum_k \phi_k\left(r_0 + \mathbf{k}^2\right)\phi_{-k}\right)$$

$$
= \prod_k \int d\phi_k \exp\left(-\frac{1}{2}\phi_k \left(r_0 + \mathbf{k}^2\right)\phi_{-k}\right)
$$

$$
= \prod_k \left(\frac{(2\pi)^d}{r_0 + \mathbf{k}^2}\right)^{1/2} = \exp\left(\frac{1}{2}\sum_k \log \chi\left(k\right)\right) \tag{66}
$$

with

$$
\chi\left(k\right) = \frac{1}{r_0 + \mathbf{k}^2}. \tag{67}
$$

It follows for the free energy

$$
F = -\frac{k_B T}{2} N \int \frac{d^d k}{(2\pi)^d} \log \chi\left(k\right) \tag{68}
$$

One can also add to the Hamiltonian an external field

$$
H\left[\phi\right] \to H\left[\phi\right] - \int d^d k h\left(k\right)\phi\left(k\right) \tag{69}
$$

Then it is easy to determine the correlation function

$$
\chi\left(k\right) = \langle \phi_k \phi_{-k}\rangle - \langle \phi_k\rangle \langle \phi_{-k}\rangle \tag{70}
$$

via

$$
\begin{aligned}
\left.\frac{\delta \log Z}{\delta h_k \delta h_{-k}}\right|_{h\to 0} &= \frac{\delta}{\delta h_k}\frac{1}{Z}\int D\phi \phi_k e^{-\beta H_{\text{eff}}[\phi]} \\
&= \frac{1}{Z}\int D\phi \phi_k \phi_{-k} e^{-\beta H_{\text{eff}}[\phi]} - \frac{\left(\int D\phi \phi_k e^{-\beta H_{\text{eff}}[\phi]}\right)^2}{Z^2} \\
&= \chi\left(k\right)
\end{aligned} \tag{71}
$$

This can again be done explicitly for the case with $u = 0$:

$$
\begin{aligned}
Z\left[h\right] &= \int D\phi \exp\left(-\frac{1}{2}\int d^d k \phi_k \left(a + b\mathbf{k}^2\right)\phi_{-k} + \int d^d k h\left(k\right)\phi_k\right) \\
&= Z\left[0\right]\exp\left(\frac{1}{2}\int d^d k h_k \chi\left(k\right) h_{-k}\right)
\end{aligned} \tag{72}
$$

Performing the second derivative of $\log Z$ gives indeed $\langle \phi_k \phi_{-k}\rangle = \frac{1}{r_0 + \mathbf{k}^2}$. Thus, we obtain as expected

$$
\chi\left(k\right) = \frac{\delta \phi_k}{\delta h_{-k}}. \tag{73}
$$

Let us analyze the specific heat related to the free energy

$$
F = -\frac{k_B T}{2} N \int d^d k \log \chi\left(k\right) \tag{74}
$$

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It holds for the singular part of the specific heat

$$C \sim -\frac{\partial^2 F}{\partial r_0^2} \sim \int d^d k \chi\left(k\right)^2 \sim \int \frac{k^{d-1} dk}{\left(\xi^{-2}+k^2\right)^2} \sim \xi^{4-d} \tag{75}$$

Thus, as $\xi \to \infty$ follows that there is no singular (divergent) contribution to the specific heat if $d > 4$ just as we found in the Landau theory. However, for $d < 4$ the specific heat diverges and we obtain a behavior different from what Landau theory predicted.

Another way to see this is to study the role of inhomogeneous fluctuations as caused by the

$$H_{inh} = \frac{1}{2} \int d^d r \left(\nabla \phi\right)^2 \tag{76}$$

Consider a typical variation on the scale $\nabla \phi \sim \sqrt{\frac{-r_0}{u}} \xi^{-1}$ and integrate those over a volume of size ξ^d gives

$$H_{inh} \sim \xi^{d-2} \frac{r_0}{u} \sim \frac{1}{u} \xi^{d-4} \tag{77}$$

Those fluctuations should be small compared to temperature in order to keep mean field theory valid. If their energy is large compared to $k_B T$ they will be rare and mean field theory is valid. Thus we obtain again that mean field theory breaks down for $d < 4$. This is called the Ginzburg criterion. Explicitly this criterion is

$$\xi^{-1} > \left(u k_B T\right)^{\frac{1}{4-d}} . \tag{78}$$

Note, if b is large for some reason, fluctuation physics will enter only very close to the transition. This is indeed the case for many so called conventional superconductors.

1.4 Scaling laws

A crucial observation of our earlier results of second order phase transitions was the divergence of the correlation length

$$\xi\left(T \to T_c\right) \to \infty. \tag{79}$$

This divergency implies that at the critical point no characteristic length scale exists, which is in fact an important reason for the emergence of the various power laws. Using h as a dimensionless number proportional to an external field and

$$r = \frac{T - T_c}{T_c} \tag{80}$$

as dimensionless measure of the distance to the critical point the various critical exponents are:

$$\xi\left(r, h = 0\right) \quad \sim \quad r^{-\nu}$$

$$
\begin{aligned}
\phi\left(r, h=0\right) &\sim |r|^{\beta} \\
\phi\left(r=0, h\right) &\sim h^{1/\delta} \\
\chi\left(r, h=0\right) &\sim r^{-\gamma} \\
C\left(r, h=0\right) &\sim r^{-\alpha} \\
\chi\left(x \to \infty, r=0\right) &\sim x^{2-d-\eta}.
\end{aligned}
\tag{81}
$$

where d is the spatial dimensionality. The values of the critical exponents for a number of systems are given in the following table

exponent	mean field	$d=2$, Ising	$d=3$, Ising
α	0	0	0.12
β	$\frac{1}{2}$	$\frac{1}{8}$	0.31
γ	1	$\frac{7}{4}$	1.25
ν	$\frac{1}{2}$	1	0.64
δ	3	15	5.0
η	0	$\frac{1}{4}$	0.04

It turn out that a few very general assumptions about the *scaling* behavior of the correlation function $\chi\left(q\right)$ and the free energy are sufficient to derive very general relations between these various exponents. Those relations are called *scaling laws*. We will argue that the fact that there is no typical length scale characterizing the behavior close to a second order phase transition leads to a powerlaw behavior of the singular contributions to the free energy and correlation function. For example, consider the result obtained within Landau theory

$$
\chi\left(q, r\right) = \frac{1}{r + q^2}
\tag{82}
$$

where we eliminated irrelevant prefactors. Rescaling all length r of the system according to $x \to x/b$, where b is an arbitrary dimensionless number, leads to $k \to kb$. Obviously, the mean field correlation function obeys

$$
\chi\left(q, r\right) = b^2 \chi\left(bq, b^2 r\right).
\tag{83}
$$

Thus, upon rescaling ($k \to kb$), the system is characterized by a correlation function which is the same up to a prefactor and a readjustment of the distance from the critical point. In what follows we will generalize this expression and assume that even beyond the mean field theory of Landau a similar relationship holds

$$
\chi\left(q, r\right) = b^{2-\eta} \chi\left(bq, b^y r\right).
\tag{84}
$$

The mean field theory is obviously recovered if $y=2$ and $\eta=0$. Since b is arbitrary, we can for example chose $tb^y = 1$ implying $b = t^{-\frac{1}{y}}$ and we obtain directly from our above ansatz

$$
\chi\left(q, t\right) = r^{-\frac{2-\eta}{y}} \chi\left(qr^{-\frac{1}{y}}, 1\right).
\tag{85}
$$

By definition, the correlation length is the length scale which characterizes the momentum variation of $\chi(q,r)$ i.e. $\chi(q,r) \sim f(q\xi)$, which leads to $\xi \sim r^{-\frac{1}{y}}$ and we obtain

$$\nu = y^{-1}. \tag{86}$$

The exponent y of our above ansatz for $\chi(q,r)$ is therefore directly related to the correlation length exponent. This makes it obvious why it was necessary to generalize the mean field behavior. $y = 2$ yields the mean field value of ν. Next we consider $r = 0$ and chose $bq = 1$ such that

$$\chi(q,r=0) = \frac{1}{q^{2-\eta}}\chi(1,0) \tag{87}$$

which gives

$$\chi(x,r=0) = \int \frac{d^dq}{(2\pi)^d}\chi(q,r=0)\,e^{ikx} \sim \int dq\, e^{ikx}\frac{q^{d-1}}{q^{2-\eta}} \tag{88}$$

substituting $z = kx$ gives

$$\chi(x,r=0) \sim x^{2-d-\eta}. \tag{89}$$

Thus, the exponent η of Eq.84 is indeed the same exponent as the one given above. This exponent is often called anomalous dimension and characterizes the change in the powerlaw decay of correlations at the critical point (and more generally for length scales smaller than ξ). Thus we can write

$$\chi(q,r) = b^{2-\eta}\chi\left(bq, b^{\frac{1}{\nu}}r\right). \tag{90}$$

Similar to the correlation function can we also make an assumption for the free energy density

$$f(r,h) = b^{-d}F\left(rb^y, hb^{y_h}\right). \tag{91}$$

The prefactor b^{-d} is a simple consequence of the fact that an extensive quantity changes upon rescaling of length with a corresponding volume factor. Using $y = \nu^{-1}$ we can again use $tb^y = 1$ and obtain

$$f(r,h) = r^{d\nu}F\left(1, hr^{-\nu y_h}\right). \tag{92}$$

This enables us to analyze the specific heat at $h = 0$ as

$$C \sim \frac{\partial^2 F(r,0)}{\partial r^2} \sim r^{d\nu-2} \tag{93}$$

which leads to

$$\alpha = 2 - d\nu. \tag{94}$$

This is a highly nontrivial relationship between the spatial dimensionality, the correlation length exponent and the specific heat exponent. It is our first scaling

law. Interestingly, it is fulfilled in mean field (with $\alpha = 0$ and $\nu = \frac{1}{2}$) only for $d = 4$.

The temperature variation of the order parameter is given as

$$\phi(r) \sim \left. \frac{\partial f(r,h)}{\partial h} \right|_{h \to 0} \sim r^{\nu(d - y_h)} \tag{95}$$

which gives

$$\beta = \nu(d - y_h) = 2 - \alpha - \nu y_h \tag{96}$$

This relationship makes a relation between y_h and the critical exponents just like y was related to the exponent ν. Within mean field

$$y_h = 3 \tag{97}$$

Alternatively we can chose $hb^{y_h} = 1$ and obtain

$$f(r,h) = h^{\frac{d}{y_h}} f\left(rh^{-\frac{1}{\nu y_h}}, 0\right) \tag{98}$$

This gives for the order parameter at the critical point

$$\phi(r = 0, h) \sim \frac{\partial f(r = 0, h)}{\partial h} \sim h^{\frac{d}{y_h} - 1} \tag{99}$$

and gives $\frac{1}{\delta} = \frac{d}{y_h} - 1$. One can simplify this to

$$\delta = \frac{y_h}{d - y_h} = \frac{2 - \alpha - \beta}{\beta} \tag{100}$$

and yields

$$\beta(1 + \delta) = 2 - \alpha \tag{101}$$

Note, the mean field theory obeys $\delta = \frac{y_h}{y_h - d}$ only for $d = 4$. whereas $\delta = \frac{2 - \alpha - \beta}{\beta}$ is obeyed by the mean field exponents for all dimensions. This is valid quite generally, scaling laws where the dimension, d, occurs explicitly are fulfilled within mean field only for $d = 4$ whereas scaling laws where the dimensionality does not occur are valid more generally.

The last result allows us to rewrite our original ansatz for the free energy

$$f(r,h) = b^{(2-\alpha)\nu^{-1}} f\left(rb^{\frac{1}{\nu}}, hb^{\frac{\beta\delta}{\nu}}\right). \tag{102}$$

such that $tb^{\frac{1}{\nu}} = 1$ leads to

$$f(r,h) = f^{2-\alpha} r\left(1, ht^{-\beta\delta}\right) \tag{103}$$

We next analyze how the susceptibility diverges at the critical point. It holds

$$\chi \sim \left. \frac{\partial^2 f(r,h)}{\partial h^2} \right|_{h \to 0} \sim r^{2 - \alpha - 2\beta\delta} \tag{104}$$

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which leads to

$$\gamma = \alpha - 2 + 2\beta\delta \tag{105}$$

which is yet another scaling relation.

The last scaling law follows from the fact that the correlation function $\chi(q, r)$ taken at $q = 0$ equals the susceptibility χ just analyzed. This gives

$$\chi(r) = b^{2-\eta}\chi(rb^y) \tag{106}$$

and choosing again $rb^y = 1$ yields

$$\chi(r) = r^{-\nu(2-\eta)} \tag{107}$$

such that

$$\gamma = \nu(2 - \eta). \tag{108}$$

To summarize, we have identified all the exponents in the assumed scaling relations of $F(t, h)$ and $\chi(q, t)$ with critical exponents (see Eqn.90 and102). In addition we have four relationships the six exponents have to fulfill at the same time which are collected here:

$$\begin{aligned}
\alpha &= 2 - d\nu \\
\beta(1 + \delta) &= 2 - \alpha \\
2\beta\delta - \gamma &= 2 - \alpha \\
\gamma &= \nu(2 - \eta)
\end{aligned} \tag{109}$$

One can easily check that the exponents of the two and three dimensional Ising model given above indeed fulfill all these scaling laws. If one wants to calculate these exponents, it turns out that one only needs to determine two of them, all others follow from scaling laws.

1.5   Fast and slow variables

The divergency that is related the Ginzburg criterion for $d \leq 4$ is caused by long wave length, i.e. the $k \to 0$ behavior of the momentum integral. One suspicion could be that only long wave length are important for an understanding of this problem. However, this is not consistent with the scaling concept, where the rescaling parameter was always assumed to be arbitrary. In fact it fluctuations on *all* length scales are crucial close to a critical point. This is on the one hand a complication, on the other hand one can take advantage of this beautiful property. Consider for example the scaling properties of the correlation function

$$\chi(q, r) = b^{2-\eta}\chi\left(bq, rb^{\frac{1}{\nu}}\right). \tag{110}$$

Repeatedly we chose $rb^{\frac{1}{\nu}} = 1$ such that $b = r^{-\nu} \to \infty$ as one approaches the critical point. However, if this scaling property (and the corresponding scaling relation for the free energy) are correct for generic b (of course only if the system

is close to T_c) one might analyze a rescaling for b very close to 1 and infer the exponents form this more "innocent" regime. If we obtain a scaling property of $\chi(q,r)$ it simply doesn't matter how we determined the various exponents like ν, η etc.

This, there are two key ingredients of the renormalization group. The first is the assumption that scaling is a sensible approach, the second is a decimation procedure which makes the scaling transformation $x \to x/b$ explicit for $b \simeq 1$. A convenient way to do this is by considering $b = e^l$ for small l. Lets consider a field variable

$$\phi(\mathbf{k}) = \int d^d x \exp(i\mathbf{k}\cdot\mathbf{x}) \phi(\mathbf{x}) \tag{111}$$

Since there is an underlying smallest length-scale a ($\simeq$interatomic spacing), no waves with wave number larger than a given upper cut off $\Lambda < 2\pi/a$ should occur. For our current analysis the precise value of Λ will be irrelevant, what matters is that such a cut off exists. Thus, be observe that $\phi(k) = 0$ if $k > \Lambda$.

We need to develop a scheme which allows us to explicitly rescale length or momentum variables. How to do this goes back to the work of Leo Kadanoff and Kenneth G. Wilson in the early 70$^{\text{th}}$ of the last century. The idea is to divide the typical length variations of $\phi(k)$ into short and long wave length components

$$\phi(k) = \begin{cases} \phi^<(k) & 0 < k \leq \Lambda/b \\ \phi^>(k) & \Lambda/b < k \leq \Lambda \end{cases}. \tag{112}$$

If one now eliminates the degrees of freedoms $\phi^>$ one obtains a theory for $\phi^<$ only

$$\exp\left(-H'\left[\phi^<\right]\right) = \int D\phi^> \exp\left(-H\left[\phi^<, \phi^>\right]\right). \tag{113}$$

The momenta in $H'[\phi^<]$ are confined to the smaller region $0 < k \leq \Lambda/b$. We can now rescale simply according to

$$k' = bk \tag{114}$$

such that the new variable k' is restricted to the original scales $0 < k' \leq \Lambda$. The field variable is then $\phi^<(k'/b)$ and will conveniently be called

$$\phi'(k') = b^{-\rho}\phi^<(k'/b) \tag{115}$$

where the prefactor $b^{-\rho}$ is only introduced for later convenience to be able to keep the prefactor of the k^2 term in the Hamiltonian the same. The *renormalized* Hamiltonian is then determined by $H'[\phi']$.

In practice we start for example from a theory of the type Eq.24 and obtain a renormalized Hamiltonian

$$H(r,u) \to H'(r(l), u(l)). \tag{116}$$

If one now analyzes the so-called flow equation of the parameters $r(l)$, $u(l)$ etc. there are a number of distinct cases. The most interesting one occurs if one approaches a fixed point where $r(l \to \infty) = r^*$, $u(l \to \infty) = u^*$ etc. If this is the case the low energy behavior of the system is identical for all initial values which reach the fixed point.

1.6 Scaling behavior of the correlation function:

We start from $H[\phi]$ with cut off scale Λ. The new Hamiltonian with cut off Λ/b, which results from the shell integration, is then determined by

$$e^{-H'[\phi^<]} = \int D\phi^> e^{-H[\phi^<, \phi^>]}, \tag{117}$$

which is supplemented by the rescaling

$$\phi^<(k) = b^\rho \phi'(bk)$$

which yields the new Hamiltonian $H'[\phi']$ which is governed by the same cut off Λ. If one considers states with momenta with $k < \Lambda/b$, it is possible to determine the corresponding correlation function either from $H[\phi]$ or from $H'[\phi']$. Thus, we can either start from the original action:

$$\langle \phi(k_1) \phi(k_2) \rangle = \int \frac{D\phi e^{-H[\phi]}}{Z} \phi(k_1) \phi(k_2) = \chi(k_1) \delta(k_1 + k_2) \tag{118}$$

or, alternatively, use the renormalized action:

$$\begin{aligned} \langle \phi(k_1) \phi(k_2) \rangle &= \int \frac{D\phi' e^{-H'[\phi']}}{Z'} b^{2\rho} \phi'(bk_1) \phi'(bk_1) \\ &= b^{2\rho} \chi'(kb_1) \delta(bk_1 + bk_2) \\ &= b^{2\rho-d} \chi'(bk_1) \delta(k_1 + k_2) \end{aligned} \tag{119}$$

where $\chi'(bk) = \chi(bk, r(l), u(l))$ is the correlation function evaluated for H' i.e. with parameters $r(l)$ and $u(l)$ instead of the "bare" ones r and u, respectively. It follows

$$\chi(k, r, u) = b^{2\rho-d} \chi(k, r(l), u(l)) \tag{120}$$

This is close to an actual derivation of the above scaling assumption and suggests to identify

$$2\rho - d = 2 - \eta. \tag{121}$$

What is missing is to demonstrate that $r(l)$ and $u(l)$ give rise to a behavior $te^{yl} = tb^y$ of

1.7 ε-expansion of the ϕ^4-theory

We will now follow the recipe outlined in the previous paragraphs and explicitly calculate the functions $r(l)$ and $u(l)$. It turns out that this can be done in a controlled fashion for spatial dimensions close to $d = 4$ and we therefore perform an expansion in $\varepsilon = 4 - d$. First we consider the free part of the Hamiltonian given by:

$$H_0(\phi) = \frac{1}{2} \int^\Lambda \frac{d^d k}{(2\pi)^d} \chi_0^{-1}(k) \phi(k) \cdot \phi(-k) \tag{122}$$

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where

$$\chi_0^{-1}(k) = r_0 + k^2. \tag{123}$$

Incidentally, if we want to determine the correlation function, it follows from usual Gaussian integration:

$$\langle \phi(k)\phi(-k) \rangle = \frac{\int D\phi\phi(k)\phi(-k)e^{-H_0}}{\int D\phi e^{-H_0}} = \chi_0(k). \tag{124}$$

Here we used $\int d\phi \phi^2 \exp\left(-\frac{1}{2\chi_0}\phi^2\right) = \sqrt{2\pi}\chi_0^{3/2}$ and $\int d\phi \exp\left(-\frac{1}{2\chi_0}\phi^2\right) = \sqrt{2\pi}\chi_0^{1/2}$.

Integrating out states in the momentum shell between Λ/b and Λ with $b = e^l$ yields an additive correction δF to the free energy and we are left with the effective action of states $\phi^<(q)$ with momenta smaller than Λ/b:

$$H_0'(\phi^<) = \frac{1}{2}\int_0^{\Lambda/b} \frac{d^d k}{(2\pi)^d}\left(r + k^2\right)\phi^<(k)\cdot\phi^<(-k) \tag{125}$$

In order to recover the original form of the action we finally rescale momentum and temperature via: $\mathbf{k}' = b\mathbf{k}$ and $\phi'(k') = Z_\phi^{1/2}\phi^<(k)$ with $Z_\phi = b^{-2\rho}$. Choosing $2\rho = d + 2$ gives finally the renormalized action:

$$H_0'(\phi') = \frac{1}{2}\int_0^{\Lambda} \frac{d^d k'}{(2\pi)^d}\left(r(l) + k'^2\right)\phi'(k')\cdot\phi'(-k') \tag{126}$$

The renormalized parameter $r(l)$ obeys the following equation:

$$\frac{dr(l)}{dl} = 2r(l) \tag{127}$$

Next we consider the quartic term

$$H_{\text{int}} = \frac{u}{4}\int d^d k_1 d^d k_2 d^d k_3 \phi(k_1)\phi(k_2)\phi(k_3)\phi(-k_1 - k_2 - k_3) \tag{128}$$

which couples $\phi^>$ and $\phi^<$. If all three momenta are inside the inner shell, we can easily perform the rescaling and find

$$H_{\text{int}}' = \frac{ub^{4\rho - 3d}}{4}\int d^D k_1' d^D k_2' d^D k_3' \phi(k_1')\phi(k_2')\phi(k_3')\phi(-k_1' - k_2' - k_3') \tag{129}$$

which gives with the above result for ρ:

$$4\rho - 3d = 4 - d \tag{130}$$

yielding

$$u(l) = ue^{(4-d)l}. \tag{131}$$

which is equivalent:

$$\frac{du(l)}{dl} = (4 - d)u(l). \tag{132}$$

Thus, at tree level, the Gaussian fixed point $u^* = 0$ is unstable if $d < 4$. If however some of the momenta in H_{int} are in the outer shell and others have Fourier modes inside Λ/b we need to analyze the coupling between them. Integrating out of states in the momentum shell between Λ/b and Λ can be performed using

$$
\begin{aligned}
\exp\left(-H'(\phi^<)\right) &\propto \exp\left(-H_0'(\phi^<)\right) \left\langle \exp\left(-H_{int}(\phi^<, \phi^>)\right) \right\rangle_> \\
&\equiv \exp\left(-H_0'(\phi^<) - \delta H'(\phi^<)\right)
\end{aligned}
\tag{133}
$$

where the average $\langle \cdots \rangle_>$ is with respect the spin excitations, $\phi^>$, with momenta between Λ/b and Λ that are characterized by $H_0'(\phi^>)$. Within the one loop approximation, it is useful to use the cumulant expansion:

$$
\delta H' = \langle H_{int} \rangle_> - \frac{1}{2}\left(\langle H_{int}^2 \rangle_> - \langle H_{int} \rangle_>^2\right) + \cdots.
\tag{134}
$$

Explicitly it holds for the correction term to the action:

$$
\begin{aligned}
H_{int}(\phi^<, \phi^>) &= \frac{u}{4} \int^\Lambda \frac{d^d k_1}{(2\pi)^d} \cdots \int^\Lambda \frac{d^d k_4}{(2\pi)^d} (\phi^<(k_1) + \phi^>(k_1)) \cdot (\phi^<(k_2) + \phi^>(k_2)) \\
&\quad (\phi^<(k_3) + \phi^>(k_3)) \cdot (\phi^<(k_4) + \phi^>(k_4)) \delta_{k_1+k_2+k_3+k_4}.
\end{aligned}
\tag{135}
$$

Averaging with respect to the $\phi^>(k)$ excitations yields non-vanishing contributions only if the number of $\phi^>(k)$ fields in the corresponding term is even. The case with zero $\phi^>(k)$ was already discussed above. It is referred to as the tree level contribution to the interaction term. The term with four $\phi^>(k)$ fields yields a constant which renormalizes the free energy on the two loop level. Finally, there are in case of $N = 3$ altogether 10 contributions with two $\phi^>(k)$ fields. Here the remaining two field carry momentum $|\mathbf{k}| < \Lambda/b$. These terms renormalize the free Hamiltonian. In the case of an $O(N)$ symmetric vector field, there are altogether $2(N + 2)$ contributions of this kind, yielding

$$
\begin{aligned}
\delta H_0'(\phi^<) &= 4(N+2)\frac{u}{4} \int^> \frac{d^d q}{(2\pi)^d} \langle \phi(q) \phi(-q) \rangle_> \frac{1}{2} \int^{\Lambda/b} \frac{d^d k}{(2\pi)^d} \phi^<(k) \cdot \phi^<(-k) \\
&= (N+2)u \int^> \frac{d^d q}{(2\pi)^d} G(p) \frac{1}{2} \int^{\Lambda/b} \frac{d^d k}{(2\pi)^d} \phi^<(k) \cdot \phi^<(-k),
\end{aligned}
\tag{136}
$$

where $\int^> d^d q$ denotes a momentum integration with $|\mathbf{q}|$ between Λ/b and Λ.

Considering the second term on the right hand side of Eq.134, which is of order u^2, at the one loop level it is sufficient to consider renormalizations of the interaction part of the action. All renormalizations of the free part will contain two closed loops and are beyond the present single loop approximation. There are altogether $8(N + 8)$ combinations to contract spin fields $\phi^<(k)$ leaving four $\phi^>(k)$ fields which can finally be expressed as:

$$
\delta H_{int}'(\phi^<) = -\frac{1}{2}8(N+8)\left(\frac{u}{4}\right)^2 \int^{\Lambda/b} \frac{dk_1}{(2\pi)^d} \cdots \int^{\Lambda/b} \frac{dk_4}{(2\pi)^d}
$$

$$\times \quad \phi^{<}(k_1) \cdot \phi^{<}(k_2)\phi^{<}(k_3) \cdot \phi^{<}(k_4)$$

$$\times \quad \int^{>} \frac{d^d p}{(2\pi)^d} G(p)G(q_1 + q_2 - p)\, \delta_{k_1+k_2+k_3+k_4} \qquad (137)$$

From these considerations we can finally obtain the renormalization group equations for the correlation length and coupling constant within the one loop approximation,

$$
\begin{aligned}
r' &= e^{2l}r + (N+2)u \int^{>} \frac{d^d q}{(2\pi)^d} G(p) \\
u' &= e^{(4-d)l}u - (N+8)u^2 \int^{>} \frac{d^d q}{(2\pi)^d} G(p)G(-p).
\end{aligned}
\qquad (138)
$$

The key difference to a straightforward perturbation theory is, that the momentum integration is restricted to the shell with radius between Λ/b and Λ. This avoids all the complications of a direct perturbation theory where a divergency in u' would result from the lower limit of the integration (long wave lengths). Integrals of the type

$$I = \int_{\Lambda/b < k < \Lambda} \frac{d^d k}{(2\pi)^d} f(k) = K_d \int_{\Lambda e^{-l}}^{\Lambda} k^{d-1} f(k)\, dk \qquad (139)$$

where the integration over angles yields $K_d = \frac{2}{(2\sqrt{\pi})^d \Gamma(d/2)}$, i.e. $K_2 = \frac{1}{2\pi}$, $K_3 = \frac{1}{2\pi^2}$, or $K_4 = \frac{1}{8\pi^2}$. For small l follows:

$$I \simeq K_d \Lambda^{d-1} f(\Lambda) \left(\Lambda - \Lambda e^{-l}\right) \simeq K_d \Lambda^d f(\Lambda)\, l \qquad (140)$$

It holds therefore

$$
\begin{aligned}
r' &= (1 + 2l)\,r + \frac{(N+2)\,K_d \Lambda^d}{r + \Lambda^2} ul \\
u' &= (1 + \varepsilon l)\,u - \frac{(N+8)\,K_d \Lambda^d}{(r + \Lambda^2)^2} u^2 l,
\end{aligned}
\qquad (141)
$$

which is due to the small-l limit conveniently written as a differential equation

$$
\begin{aligned}
\frac{dr}{dl} &= 2r + \frac{(N+2)\,K_d \Lambda^d}{r + \Lambda^2} u, \\
\frac{du}{dl} &= \varepsilon u - \frac{(N+8)\,K_d \Lambda^d}{(r + \Lambda^2)^2} u^2.
\end{aligned}
\qquad (142)
$$

Before we proceed, we introduce more convenient variables

$$r \rightarrow \frac{r}{\Lambda^2} \qquad (143)$$

$$u \rightarrow K_d \Lambda^{d-4} u \qquad (144)$$

which are dimensionless and obtain the differential equations

$$\frac{dr}{dl} = 2r + \frac{(N+2)\,u}{1+r}$$
$$\frac{du}{dl} = \varepsilon u - \frac{(N+8)u^2}{(1+r)^2}. \tag{145}$$

The system has indeed a fixed point (where $\frac{dr}{dl} = \frac{du}{dl} = 0$) determined by

$$\varepsilon = \frac{(N+8)\,u^*}{(1+r^*)^2}$$
$$2r^* = -\frac{(N+2)\,u^*}{1+r^*} \tag{146}$$

This simplifies at leading order in ε to

$$u^* = \frac{\varepsilon}{N+8} \text{ or } 0$$
$$r^* = -\frac{(N+2)}{2}u^* \tag{147}$$

If the system reaches this fixed point it will be governed by the behavior in its immediate vicinity, allowing us to linearize the flow equation in the vicinity of the fixed point, i.e. for small

$$\delta r = r - r^*$$
$$\delta u = u - u^*. \tag{148}$$

Consider first the fixed point with $u^* = r^* = 0$ gives

$$\frac{d}{dl}\begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & \varepsilon \end{pmatrix}\begin{pmatrix} \delta r \\ \delta u \end{pmatrix} \tag{149}$$

with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \varepsilon$. Both eigenvalues are positive for $\varepsilon > 0$ ($d < 4$) such that there is no scenario under which this fixed point is ever governing the low energy physics of the problem.

Next we consider $u^* = \frac{\varepsilon}{9}$ and $r^* = -\frac{\varepsilon}{6}$. It follows

$$\frac{d}{dl}\begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 - \frac{N+2}{N+8}\varepsilon & N+2+\frac{(N+2)^2}{2(N+8)}\varepsilon \\ 0 & -\varepsilon \end{pmatrix}\begin{pmatrix} \delta r \\ \delta u \end{pmatrix} \tag{150}$$

with eigenvalues

$$y = 2 - \frac{N+2}{N+8}\varepsilon$$
$$y' = -\varepsilon \tag{151}$$

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the corresponding eigenvectors are

$$
\begin{aligned}
e &= (1,0) \\
e' &= \left(-\frac{3\,(N+2)\,(N+6)}{4\,(N+8)}+\frac{\varepsilon}{8},1\right)
\end{aligned}
\tag{152}
$$

Thus, a variation along the e-direction (which is varying r) causes the system to leave the fixed point (positive eigenvalue), whereas it will approach the fixed point if

$$
(r,u) \sim e'
\tag{153}
$$

this gives

$$
r_c\,(u) = u\left(-\frac{3\,(N+2)\,(N+6)}{4\,(N+8)}+\frac{\varepsilon}{8}\right)
\tag{154}
$$

which defines the *critical surface* in parameter space. If a system is on this surface it approaches the fixed point. If it is slightly away, the quantity

$$
t = r - r_c\,(u)
\tag{155}
$$

is non-zero and behaves as

$$
t\,(l) = t e^{yl} = t b^y.
\tag{156}
$$

The flow behavior for large l is only determined by the value of t which is the only scaling variable, which vanishes at the critical point. Returning now to the initial scaling behavior of the correlation function we can write explicitly

$$
\chi\,(k,t) = b^2 \chi\,(k,t b^y)
\tag{157}
$$

comparing this with $\chi\,(q,t) = b^{2-\eta}\chi\left(bq,t b^{\frac{1}{\nu}}\right)$ gives immediately the two critical exponents

$$
\begin{aligned}
\eta &= \mathcal{O}\left(\varepsilon^2\right) \\
\nu &\simeq \frac{1}{2}+\frac{N+2}{N+8}\frac{\varepsilon}{4}.
\end{aligned}
\tag{158}
$$

A systematic improvement of these results occurs if one includes higher order terms of the εexpansion. Thus, the renormalization group approach is a very powerful tool to analyze the highly singular perturbation expansion of the ϕ^4-theory below its upper critical dimension. How is it possible that one can obtain so much information by essentially performing a low order expansion in u for a small set of high energy degrees of freedom? The answer is in the power of the scaling concept. We have assumed that the form $\chi\,(q,t) = b^{2-\eta}\chi\left(bq,t b^{\frac{1}{\nu}}\right)$ which we obtained for very small deviations of b from unity is valid for all b. If for example the value of ν and η would change with l there would be no way that we could determine the critical exponents from such a procedure. If scaling does not apply, no critical exponent can be deduced from the renormalization group.some quantity t which vanishes at the phase transition. To see this is easier if one performs the calculation explicitly.

2 Quantum critical behavior

Part III

Critical behavior in disordered systems

3 Random mass disorder

Next we wish to describe a disordered system. Imagine a spin system where the exchange interaction $J_{i,j}$ between spins is affected by disorder and fluctuates from site to site. In the continuum's description this amounts to replacing r_0 by a spatially varying function:

$$r_0 \to r_0 + \delta r\left(x\right).\tag{159}$$

One could now try to solve the problem for a given configuration of $\delta r\left(x\right)$ and try to deduce information about the system. If we are only interested in averaged quantities, it makes sense to define a distribution function of the $\delta r\left(x\right)$ and analyze averaged quantities.

References

[1] uu