Theorie der Kondensierten Materie II SS 2017

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1. Scattering amplitude: (40 Punkte)

Recall the scattering problem in quantum mechanics. Scattering of a plane wave $e^{i\vec{k}\vec{r}}$ on a static, one-particle potential $V(\vec{r})$ is described by the wave function

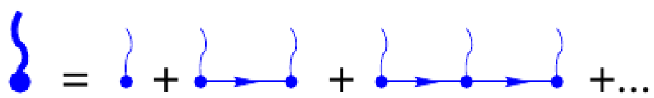
$$\psi_{\vec{k}}(\vec{r}) = e^{i\vec{k}\vec{r}} + \chi_{\vec{k}}(\vec{r}),$$

where $\chi_{\vec{k}}(\vec{r})$ has the form of a spherical wave

$$\chi_{\vec{k}}(\vec{r}) = f(\vec{k}, k\vec{n})\frac{e^{ik|\vec{r}|}}{|\vec{r}|}, \quad |\vec{r}| \to \infty, \quad \vec{n} = \frac{\vec{r}}{|\vec{r}|}, \quad k = |\vec{k}|.$$

The function $f(\vec{k}, k\vec{n})$ is known as the scattering amplitude.

(a) Show that the scattering amplitude can be represented by a series of diagrams shown in the Figure.



Which expressions correspond to the elements of the graphs?
Hint: use the momentum representation.

(b) Derive the relation

$$f(\vec{k}_1, \vec{k}_2) = -\frac{m}{2\pi\hbar^2}F(\vec{k}_1, \vec{k}_2); \quad F(\vec{k}_1, \vec{k}_2) = V(\vec{k}_2 - \vec{k}_1) + \int \frac{d^3q}{(2\pi)^3}\frac{V(\vec{k}_2 - \vec{q})F(\vec{k}_1, \vec{q})}{\epsilon - \hbar^2q^2/(2m) + i\delta}.$$

2. Shallow well: (30 Punkte)

A “shallow well” is a potential well with the depth $U_0 \ll \hbar^2/(2ma^2)$, where a is the width (or radius) of the well. In such a potential, the energy of a bound state is much smaller than the well depth U_0, while its wave function extends over distances much greater than the well radius a.

Consider a schallow well in a D-dimensional space and find out in which case do the bound states exist.

(a) Show, that the energy of each bound state corresponds to a pole of the scattering amplitude $F(\vec{k}_1, \vec{k}_2)$ as a function of energy.

(b) Show, that bound states in shallow wells exist only for $D \leqslant 2$.

(c) Compare the results with the standard quantum-mechanical expressions.

Hints: Use the equation for $F(\vec{k}_1, \vec{k}_2)$ derived in the previous exercise. In D-dimensional space the integration measure becomes $d^D q/(2\pi)^D$. To simplify the calculations, you may replace the well potential in the equation for $F(\vec{k}_1, \vec{k}_2)$ by a δ-function.

3. Friedel oscillations (30 Punkte)

(a) For free fermions in a one-dimensional space (i.e., moving on a line) find the explicit expression for the Green's function $G_{\alpha\beta}(\epsilon; x, x')$.

(b) Repeat the calculation for the half-line $x > 0$ with the hard-wall boundary condition $\psi(x = 0) = 0$.

(c) In the latter case, show that the fermion density $n(x)$ oscillates as a function of the distance x from the boundary (the so-called Friedel oscillations). What is the period of the oscillations? Plot the resulting density $n(x)$.