Übungen zur Theoretischen Physik Fa WS 17/18

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Blatt 4
Lösungsvorschlag

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1. Nichtwechselwirkende Spins:

(a) Stationary states are characterized by the sets of quantum numbers $\{\sigma_a\}$ with the corresponding energies $E\{\sigma_a\} = -\mu \sum \sigma_a^z H = -MH$. Here M is the magnetization in the direction of the field, which we have chosen as the z axis. $\sigma_a = \pm 1$ are the z-components of the individual spins. We choose units with $k_B = 1$.

(b) The statistical sum is

$$Z = \sum_{\{\sigma_a\}} e^{-E\{\sigma_a\}/T} = \sum_{\{\sigma_a\}} \prod_{a=1}^{N} e^{-\mu H \sigma_a/T} = \prod_{a=1}^{N} \sum_{\{\sigma_a\}} e^{-\mu H \sigma_a/T} = \left[2\cosh\frac{\mu H}{T}\right]^N .$$

The free energy is the logarithm of the statistical sum

$$F = -T \ln Z = -NT \ln\left[2\cosh\frac{\mu H}{T}\right].$$

(c) The entropie:

$$S = -\left(\frac{\partial F}{\partial T}\right)_H = N \ln\left[2\cosh\frac{\mu H}{T}\right] - \frac{\mu H N}{T}\tanh\frac{\mu H}{T}.$$

The magnetization:

$$M = -\left(\frac{\partial F}{\partial H}\right)_T = \mu N \tanh\frac{\mu H}{T}.$$

The specific heat:

$$c_H = T\left(\frac{\partial S}{\partial T}\right)_H = \frac{\mu^2 H^2}{T^2 \cosh^2\frac{\mu H}{T}}.$$

The specific heat for constant magnetization can be calculated inverting the function $M(H,T)$:

$$\begin{aligned} c_M &= T\left(\frac{\partial S(T, H(M,T))}{\partial T}\right)_M \\ &= T\left(\frac{\partial S(T, H(M,T))}{\partial T}\right)_H + T\left(\frac{\partial S(T, H(M,T))}{\partial H}\right)_T \left(\frac{\partial H(M,T)}{\partial T}\right)_M = 0. \end{aligned}$$

(d) In the limit of large fields,

$$M(\mu H \gg T) \approx \mu N.$$

In the limit of small fields,

$$M(\mu H \ll T) \approx \mu^2 N \frac{H}{T}.$$

Therefore the susceptibility is given by the Curie law

$$\chi(T) = \lim_{H \to 0} \left(\frac{\partial M}{\partial H} \right)_T = \frac{\mu^2 N}{T}.$$

2. Heisenberg–Modell für 2 Gitterplätze:

According to the general rules of addtion of angular momentum, the system of two momenta can be described by a set of four quantum numbers. The two common possibilities are either (i) $\boldsymbol{s}_1^2$, $\boldsymbol{s}_2^2$, s_1^z, s_2^z, or (ii) $\boldsymbol{s}_1^2$, $\boldsymbol{s}_2^2$, $\boldsymbol{S}^2$, S^z, where $\boldsymbol{S} = \boldsymbol{s}_1 + \boldsymbol{s}_2$. Suppose we choose the first possibility. Then for fixed $\boldsymbol{s}_1^2 = s_1(s_1+1)$, $\boldsymbol{s}_2^2 = s_2(s_2+1)$, the quantities s_i^z take $2s_i + 1$ values each, so that the total number of states being $(2s_1 + 1)(2s_2 + 1)$.

For two spins 1, $s_i = 1$, we have 9 possible states.

In this problem, it is more convenient to choose the second representation, since

$$\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = \frac{1}{2} \left[S(S+1) - s_1(s_1+1) - s_2(s_2+1) \right].$$

The 9 possible states correspond to the following values (fixing $s_1 = s_2 = 1$):

$$S = 0, \quad S^z = 0, \quad \Rightarrow \boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = -2,$$

$$S = 1, \quad S^z = 0, \pm 1, \quad \Rightarrow \boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = -1,$$

$$S = 2, \quad S^z = 0, \pm 1, \pm 2, \quad \Rightarrow \boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = 1.$$

(a) In the absence of the field, the statistical sum is given by

$$Z = \sum_{\{S, S^z\}} e^{J \boldsymbol{s}_1 \cdot \boldsymbol{s}_2 / T} = 5 e^{J/T} + 3 e^{-J/T} + e^{-2J/T} = 6 \cosh J/T + 2 e^{J/T} + e^{-2J/T}.$$

The free energy is

$$F = -T \ln Z.$$

The specific heat is

$$C_V = -T \frac{\partial^2 F}{\partial T^2} = \frac{J^2}{T^3} e^{-2J/T} \left[4 \left(\frac{T}{J} - 1 \right) + 3 \left(\frac{2T}{J} - 1 \right) e^{J/T} - 5 \left(\frac{2T}{J} + 1 \right) e^{3J/T} \right]$$

(b) In the presence of the field, the statistical sum is

$$Z = \sum_{\{S,S^z\}} e^{(J\boldsymbol{s}_1\cdot\boldsymbol{s}_2+\mu H S^z)/T} =$$

$$= e^{J/T}\left(1+2\cosh\frac{\mu H}{T}+2\cosh\frac{2\mu H}{T}\right)+e^{-J/T}\left(1+2\cosh\frac{\mu H}{T}\right)+e^{-2J/T}.$$

The free energy is

$$F = -T\ln Z.$$

The magnetization is

$$M = -\frac{\partial F}{\partial H} = \frac{4\mu}{Z}\left[\cosh\frac{J}{T}\sinh\frac{\mu H}{T}+e^{J/T}\sinh\frac{2\mu H}{T}\right].$$

The susceptibility is

$$\chi(T) = \frac{4\mu^2}{Z_0 T}\left[\cosh\frac{J}{T}+2e^{J/T}\right],$$

where Z_0 is the above statistical sum in the absence of the field. This susceptibility becomes Curie-like only at high temperatures $T \gg J$.

3. Masselose relativistische Teilchen:

(a) The statistical sum of a system of non-interacting particles factorizes:

$$Z = \frac{Z_1^N}{N!}.$$

The single-particle "statistical sum" is given by (here $\hbar = c = 1$)

$$Z_1 = V\int\frac{d^2p}{(2\pi)^2}e^{-c|\boldsymbol{p}|/T} = \frac{V}{2\pi}\int_0^\infty pe^{-cp/T}dp = \frac{VT^2}{2\pi c^2}\int_0^\infty ze^{-z}dz = \frac{VT^2}{2\pi c^2}.$$

Hence, the statistical sum is

$$Z = \frac{V^N}{N!(2\pi)^N}\left(\frac{T}{c}\right)^{2N}.$$

The free energy is

$$F = -T\ln Z = -T\left\{2N\ln\frac{T\sqrt{V}}{c\sqrt{2\pi}}-\ln N!\right\} = -T\left\{2N\ln\left[\frac{T}{c}\left(\frac{V}{2\pi}\right)^{1/2}\right]-N\ln N+N\right\},$$

or

$$F = -2NT\ln\left[\frac{T}{c}\left(\frac{V}{2\pi N}\right)^{1/2}\right]-NT.$$

(b) The entropy is given by the derivative

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = -\frac{F}{T} + 2N.$$

Similarly, the pressure is

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{NT}{V}.$$

(c) The equation of state relates the internal energy of the system to temperature. The internal energy is

$$U = V \int \frac{d^2p}{(2\pi)^2} cp\theta(p_F - p) = \frac{cV}{2\pi} \int_0^{p_F} p^2 dp = \frac{cVp_F^3}{6\pi}.$$

This should be related to the particle density

$$\frac{N}{V} = \int \frac{d^2p}{(2\pi)^2} \theta(p_F - p) = \frac{1}{2\pi} \int_0^{p_F} pdp = \frac{p_F^2}{4\pi}.$$

Hence

$$p_F = \sqrt{4\pi N/V}$$

and the internal energy can be expressed as

$$U = \frac{4cN}{3} \sqrt{\frac{4\pi N}{V}}$$

The pressure can be evaluated also as

$$P = -\left(\frac{\partial U}{\partial V}\right)_{T,N} = \frac{U}{2V}.$$

Comparing with the previous result, we find the equation of state

$$PV = NT \quad \Rightarrow \quad U = 2NT.$$

4. Ideales Gas:

(a) The statistical sum of a system of non-interacting particles factorizes:

$$Z = \frac{Z_1^N}{N!}.$$

The single-particle "statistical sum" is given by (here $\hbar = c = 1$)

$$Z_1 = V \int \frac{d^3p}{(2\pi)^3} e^{-a|\boldsymbol{p}|^4/T} = \frac{V}{2\pi^2} \int_0^\infty p^2 e^{-ap^4/T} dp = \frac{VT^{3/4}}{2\pi^2 a^{3/4}} \int_0^\infty z^2 e^{-z^4} dz = \frac{V}{8\pi^2} \Gamma(3/4) \left(\frac{T}{a}\right)^{3/4}.$$

Hence, the statistical sum is

$$Z = \frac{V^N}{N!}\left[\frac{\Gamma(3/4)}{8\pi^2}\right]^N \left(\frac{T}{a}\right)^{3N/4}.$$

The free energy is

$$\begin{aligned}
F = -T\ln Z = -T&\left\{N\ln\left[V\frac{\Gamma(3/4)}{8\pi^2}\left(\frac{T}{a}\right)^{3/4}\right] - \ln N!\right\}\\
&= -T\left\{\frac{3N}{4}\ln\left[\frac{T}{a}\left(\frac{V\Gamma(3/4)}{8\pi^2}\right)^{4/3}\right] - N\ln N + N\right\},\\
&= -\frac{3}{4}NT\ln\left[\frac{T}{a}\left(\frac{V\Gamma(3/4)}{8\pi^2 N}\right)^{4/3}\right] - NT.
\end{aligned}$$

(b) The entropy is given by the derivative

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = -\frac{F}{T} + \frac{3}{4}N.$$

Similarly, the pressure is

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{NT}{V}.$$

(c) The equation of state relates the internal energy of the system to temperature. The internal energy is

$$U = V\int\frac{d^3p}{(2\pi)^3}ap^4\theta(p_F - p) = \frac{aV}{2\pi^2}\int_0^{p_F} p^6\,dp = \frac{aVp_F^7}{14\pi^2}.$$

This should be related to the particle density

$$\frac{N}{V} = \int\frac{d^3p}{(2\pi)^3}\theta(p_F - p) = \frac{1}{2\pi^2}\int_0^{p_F} p^2\,dp = \frac{p_F^3}{6\pi^2}.$$

Hence

$$p_F = \sqrt[3]{6\pi^2 N/V}$$

and the internal energy can be expressed as

$$U = \frac{3aN}{7}\left(\frac{6\pi^2 N}{V}\right)^{4/3}$$

The pressure can be evaluated also as

$$P = -\left(\frac{\partial U}{\partial V}\right)_{T,N} = \frac{4U}{3V}.$$

Comparing with the previous result, we find the equation of state

$$PV = NT \quad \Rightarrow \quad U = \frac{3}{4}NT.$$