Karlsruhe Institute of Technology

Winter-Semester 2017/18

Moderne Theoretische Physik IIIa

Statistische Physik

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Institut für Theorie der Kondensierten Materie

Do 11:30-13:00, Lehmann Raum 022, Geb 30.22

http://www.tkm.kit.edu/lehre/

Ideales Fermi-Gas, T=0

$$n = \frac{N}{V} = (2s+1) \int d\epsilon \, \nu(\epsilon) \, \theta(\mu - \epsilon)$$

Die Teilchen-Dichte n ist gegeben $\rightarrow \mu(T=0) \equiv \epsilon_F$

ϵ_F - Fermi-Energie

$$p_F = \sqrt{2m\epsilon_F}$$

$$\nu(\epsilon) = \frac{m^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}$$ Zustandsdichte

$$\mu(T=0) = \epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 n}{2s+1} \right)^{\frac{2}{3}}$$

Ideales Fermi-Gas

Entartetes Fermi-Gas $k_{\mathrm{B}}T \ll \epsilon_F$

$$Z_G(T,V,\mu) = \prod_\lambda \left[1 + e^{-\beta(\epsilon_\lambda - \mu)}\right]$$

$$\Omega(T,V,\mu) = -k_{\mathrm{B}}T \ln Z_G(T,V,\mu) = -k_{\mathrm{B}}T \sum_\lambda \ln\left[1 + e^{-\beta(\epsilon_\lambda - \mu)}\right]$$

$$\Omega(T,V,\mu) = -k_{\mathrm{B}}T(2s+1)V \int_{-\infty}^{\infty} d\epsilon\, \nu(\epsilon) \ln\left[1 + e^{-\beta(\epsilon - \mu)}\right]$$

2 x partielle Integration

$$\Omega(T,V,\mu) = -(2s+1)V \int_{-\infty}^{\infty} d\epsilon\, a(\epsilon) n_F(\epsilon)$$

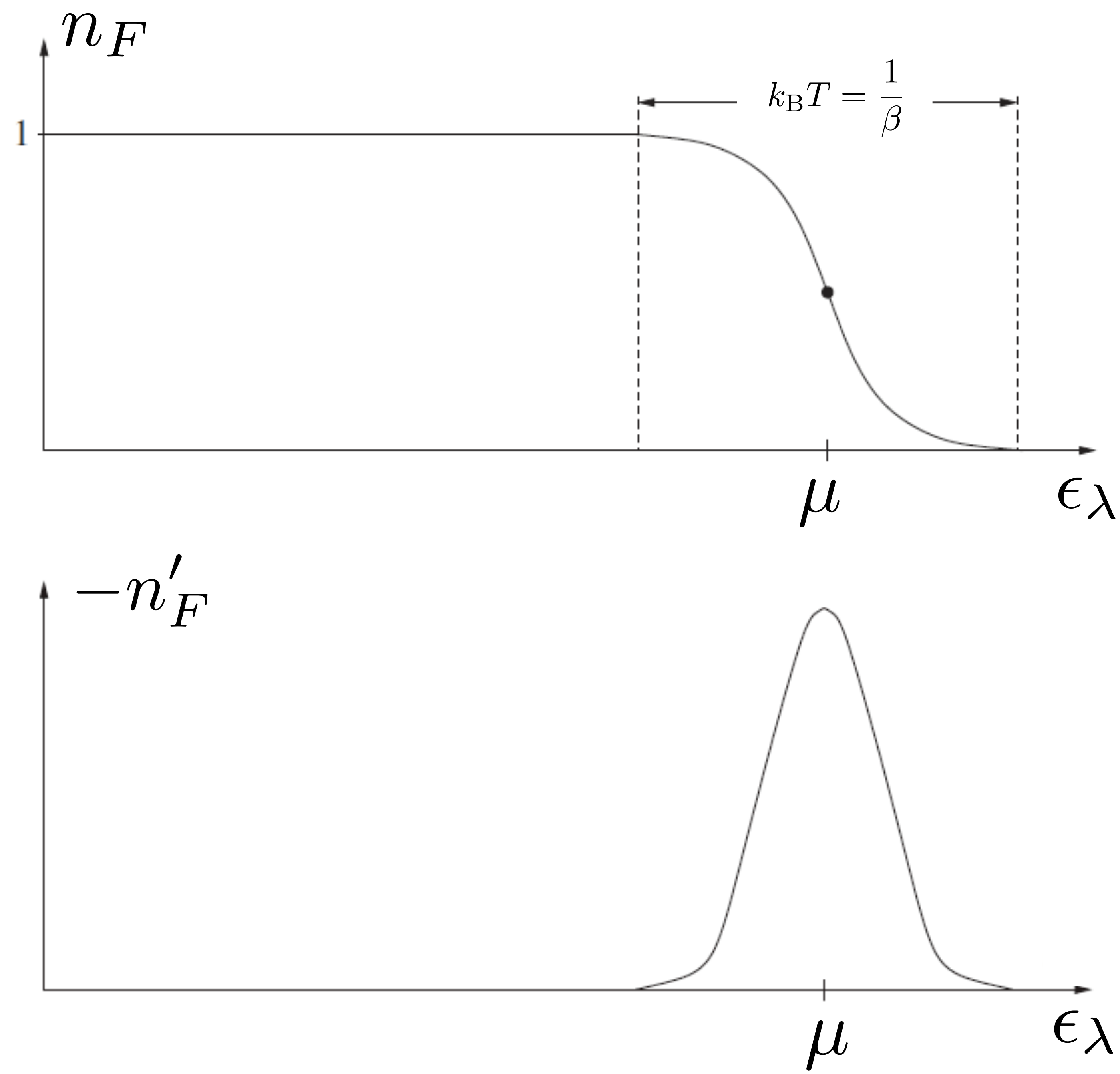
$$a(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 \nu(\epsilon_1)$$

$$\Omega(T,V,\mu) = -(2s+1)V \int_{-\infty}^{\infty} d\epsilon\, b(\epsilon)(-n_F'(\epsilon))$$

$$b(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 a(\epsilon_1)$$

Sommerfeld-Entwicklung $\quad k_B T \ll \epsilon_F$

$$-\frac{\partial n_F}{\partial \epsilon_\lambda} = \frac{1}{4k_\mathrm{B}T\cosh^2\frac{\epsilon_\lambda-\mu}{2k_\mathrm{B}T}}$$



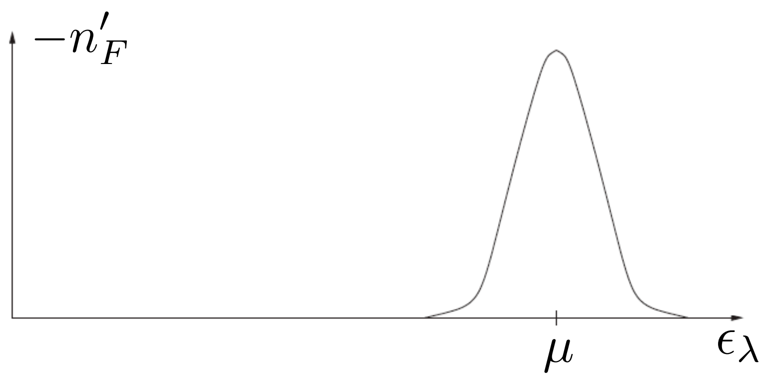
Ideales Fermi-Gas

Entartetes Fermi-Gas $k_{\mathrm{B}}T \ll \epsilon_F$

$$\Omega(T,V,\mu) = -(2s+1)V \int_{-\infty}^{\infty} d\epsilon\, b(\epsilon)(-n'_F(\epsilon))$$

$$a(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 \nu(\epsilon_1)$$

$$b(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 a(\epsilon_1)$$



für $\epsilon \approx \mu$ gilt

$$b(\epsilon) = b(\mu) + a(\mu)(\epsilon-\mu) + \frac{1}{2}\nu(\mu)(\epsilon-\mu)^2 + \ldots$$

$$\int_{-\infty}^{\infty} dx\, \frac{x^2}{\cosh^2 x} = \frac{\pi^2}{6}$$

$$\Omega(T,V,\mu) = -(2s+1)V\left[b(\mu) + \frac{\pi^2}{6}(k_{\mathrm{B}}T)^2\,\nu(\mu) + \ldots\right]$$

Sommerfeld-Entwicklung

Ideales Fermi-Gas

$$a(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 \nu(\epsilon_1)$$

Entartetes Fermi-Gas $k_{\mathrm{B}}T \ll \epsilon_F$

$$b(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 a(\epsilon_1)$$

$$\Omega(T,V,\mu) = -(2s+1)V\left[b(\mu) + \frac{\pi^2}{6}(k_{\mathrm{B}}T)^2\,\nu(\mu) + \ldots\right]$$

$$N = -\frac{\partial\Omega}{\partial\mu}\bigg|_{T,V} = (2s+1)V\left[a(\mu) + \frac{\pi^2}{6}(k_{\mathrm{B}}T)^2\,\frac{d\nu(\mu)}{d\mu} + \ldots\right]$$

$$a(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 \nu(\epsilon_1) = \frac{N(T=0,V,\mu)}{(2s+1)V}$$

$$N(T,V,\mu) = N(T=0,V,\mu) + (2s+1)V\frac{\pi^2}{6}(k_{\mathrm{B}}T)^2\,\frac{d\nu(\mu)}{d\mu} + \ldots$$

$$n(T,\mu) = n(T=0,\mu) + (2s+1)\frac{\pi^2}{6}(k_{\mathrm{B}}T)^2\,\frac{d\nu(\mu)}{d\mu} + \ldots$$

Entartetes ideales Fermi-Gas $k_{\mathrm{B}}T \ll \epsilon_F$

Das chemische Potential μ ist gegeben $\rightarrow n(T,\mu) = ?$

$$n(T,\mu) = n(T=0,\mu) + (2s+1)\frac{\pi^2}{6}(k_{\mathrm{B}}T)^2 \frac{d\nu(\mu)}{d\mu} + \ldots$$

$$\mu(T=0) = \epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{2s+1}\right)^{\frac{2}{3}} \quad \Rightarrow \quad n(T=0,\mu) = \frac{2s+1}{6\pi^2}\left(\frac{2m\mu}{\hbar^2}\right)^{\frac{3}{2}}$$

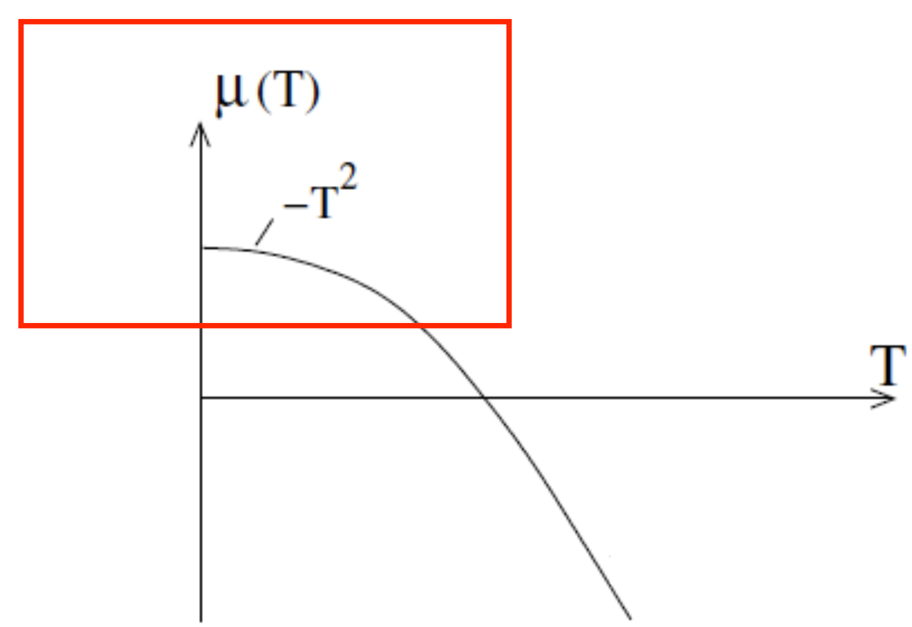
Die Teilchen-Dichte n ist gegeben $\rightarrow \mu(T,n) = ?$

$$\nu(\epsilon) = \frac{m^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}$$

$$n = \frac{2s+1}{6\pi^2}\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{\frac{3}{2}} = \frac{2s+1}{6\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\left[\mu^{3/2} + \frac{\pi^2}{8\sqrt{\mu}}(k_{\mathrm{B}}T)^2 + \ldots\right]$$

$$\epsilon_F^{3/2} = \mu^{3/2} + \frac{\pi^2}{8\sqrt{\mu}}(k_{\mathrm{B}}T)^2 + \ldots$$

$$\mu = \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_{\mathrm{B}}T}{\epsilon_F}\right)^2 + \ldots\right]$$



Ideales Fermi-Gas

$$a(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 \nu(\epsilon_1)$$

$$b(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 a(\epsilon_1)$$

Entartetes Fermi-Gas $k_\mathrm{B} T \ll \epsilon_F$

$$\Omega(T,V,\mu) = -(2s+1)V \left[b(\mu) + \frac{\pi^2}{6} (k_\mathrm{B}T)^2 \nu(\mu) + \ldots \right]$$

$$S(T,V,\mu) = -\left.\frac{\partial \Omega}{\partial T}\right|_{V,\mu} = (2s+1)V \frac{\pi^2 \nu(\mu)}{3} k_\mathrm{B}^2 T$$

kanonisch - großkanonisch: Wachsamkeit nötig $\mu = \mu(T,N)$

$$C_V = T \left(\frac{\partial S}{\partial T}\right)_{V,N} = T \left(\frac{\partial S(T,V,\mu(N,T))}{\partial T}\right)_{V,N}$$

$$C_V = T \left(\frac{\partial S(T,V,\mu)}{\partial T}\right)_{V,\mu} + T \left(\frac{\partial S(T,V,\mu)}{\partial \mu}\right)_{V,T} \left(\frac{\partial \mu(T,N)}{\partial T}\right)_{N}$$

$$C_V \approx N k_\mathrm{B} \frac{\pi^2}{2} \frac{k_\mathrm{B} T}{\epsilon_F} + O(T^3)$$

Ideales Fermi-Gas

Entartetes Fermi-Gas $k_{\mathrm{B}}T \ll \epsilon_F$

$$\Omega(T,V,\mu) = -(2s+1)V\left[b(\mu) + \frac{\pi^2}{6}\left(k_{\mathrm{B}}T\right)^2\nu(\mu) + \ldots\right]$$

$$P = -\frac{\partial\Omega}{\partial V}\bigg|_{T,\mu} = (2s+1)\left[b(\mu) + \frac{\pi^2}{6}\left(k_{\mathrm{B}}T\right)^2\nu(\mu) + \ldots\right]$$

$$P = P(T=0,V,\mu) + (2s+1)\frac{\pi^2}{6}\left(k_{\mathrm{B}}T\right)^2\nu(\mu) + \ldots$$

$$a(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 \nu(\epsilon_1) = \frac{N(T=0,\mu=\epsilon,V)}{(2s+1)V} = \frac{1}{(2\pi\hbar)^3}\cdot\frac{4}{3}\pi(2m\epsilon)^{3/2}$$

$$b(\mu) \equiv \int_{-\infty}^{\mu} d\epsilon_1 a(\epsilon_1) = \frac{2}{5}\mu\cdot\frac{N(T=0,\mu,V)}{(2s+1)V}$$

$$P(T=0,V,\mu)V = (2s+1)Vb(\mu) = \frac{2}{5}N(T=0,V,\mu)\,\mu = \frac{2}{5}Nk_{\mathrm{B}}T_F$$

Ideales Fermi-Gas

Beliebige Temperaturen

$$\Omega(T,V,\mu) = -k_{\rm B}T\sum_\lambda \ln\left[1+e^{-\beta(\epsilon_\lambda-\mu)}\right]$$

$$\nu(\epsilon) = \frac{m^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}$$

$$\begin{aligned}\Omega(T,V,\mu) &= -k_{\rm B}T(2s+1)V\int d\epsilon\,\nu(\epsilon)\,\ln\left[1+e^{-\beta(\epsilon-\mu)}\right]\\ &= -(2s+1)\,\frac{V}{\lambda_T^3}\,k_{\rm B}T f_{5/2}(z)\end{aligned}$$

$$z \equiv e^{\beta\mu} \qquad \lambda_{\rm T} = \sqrt{\frac{2\pi\hbar^2}{mk_{\rm B}T}}$$

$$f_{5/2}(z) \equiv \frac{4}{\sqrt{\pi}}\int_0^\infty dx\,x^2\ln(1+ze^{-x^2}) = \sum_{n=1}^\infty \frac{(-1)^{n+1}z^n}{n^{5/2}}$$

$$N = -\left.\frac{\partial\Omega}{\partial\mu}\right|_{T,V} = (2s+1)\frac{V}{\lambda_T^3}f_{3/2}(z)$$

$$f_{3/2}(z) \equiv \sum_{n=1}^\infty \frac{(-1)^{n+1}z^n}{n^{3/2}}$$

Ideales Fermi-Gas

Beliebige Temperaturen

$$z \equiv e^{\beta\mu} \qquad \lambda_{\mathrm{T}} = \sqrt{\frac{2\pi\hbar^2}{mk_{\mathrm{B}}T}}$$

$$\Omega(T,V,\mu) = -(2s+1)\,\frac{V}{\lambda_T^3}\,k_{\mathrm{B}}T f_{5/2}(z) \qquad f_{5/2}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n^{5/2}}$$

$$N = -\frac{\partial\Omega}{\partial\mu}\Big|_{T,V} = (2s+1)\frac{V}{\lambda_T^3} f_{3/2}(z) \qquad f_{3/2}(z) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n^{3/2}}$$

$$S(T,V,\mu) = -\frac{\partial\Omega}{\partial T}\Big|_{V,\mu} \quad \Rightarrow \quad U = -\frac{3}{2}\Omega = \frac{3}{2}PV$$

Übung

Ideales Fermi-Gas

$$z \equiv e^{\beta\mu} \qquad \lambda_{\rm T} = \sqrt{\frac{2\pi\hbar^2}{m k_{\rm B} T}}$$

Hohe Temperaturen $k_{\rm B}T \gg \epsilon_F$

Vermutung: $z \ll 1$

$$N = (2s+1)\frac{V}{\lambda_T^3} f_{3/2}(z) \approx (2s+1)\frac{V}{\lambda_T^3}\, z \qquad f_{3/2}(z) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n^{3/2}}$$

$$\Omega(T,V,\mu) = -(2s+1)\,\frac{V}{\lambda_T^3}\, k_{\rm B}T f_{5/2}(z) \approx -k_{\rm B}TN \qquad f_{5/2}(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n^{5/2}}$$

$$U = -\frac{3}{2}\Omega = \frac{3}{2}PV \quad \Rightarrow \quad U = \frac{3}{2}k_{\rm B}TN, \quad PV = k_{\rm B}TN$$

ideales klassisches Gas

Ideales Fermi-Gas

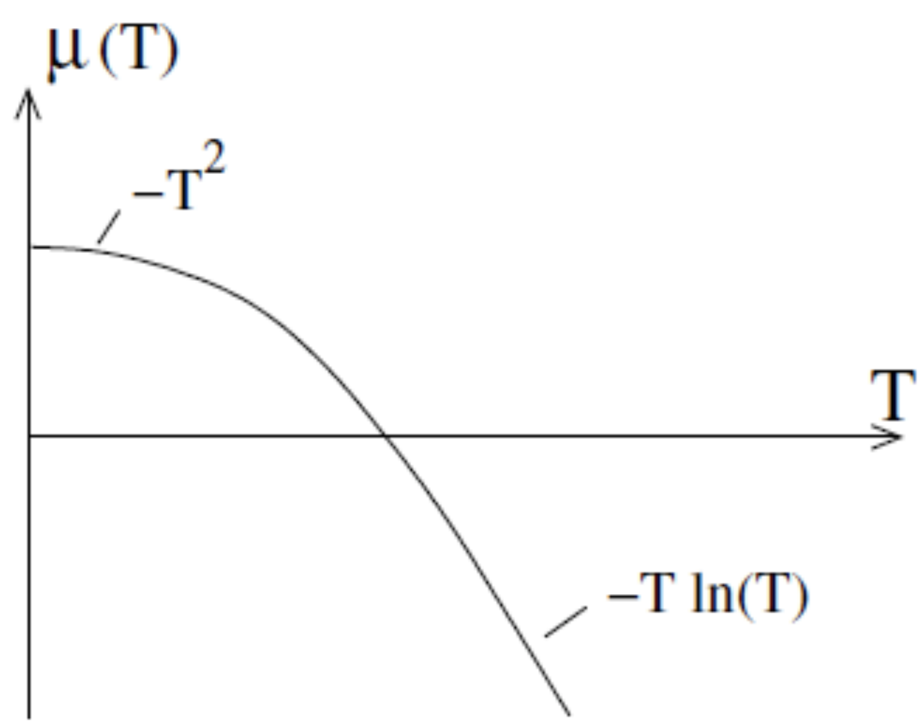
$z \equiv e^{\beta\mu}$ $\lambda_{\mathrm{T}} = \sqrt{\frac{2\pi\hbar^2}{m k_{\mathrm{B}} T}}$

Hohe Temperaturen $k_{\mathrm{B}} T \gg \epsilon_F$

Vermutung: $z \ll 1$

$$N \approx (2s+1)\frac{V}{\lambda_T^3}\, z \quad \Rightarrow \quad \mu \approx k_{\mathrm{B}} T \ln \frac{\lambda_T^3 n}{(2s+1)}$$

$n \equiv \frac{N}{V}$



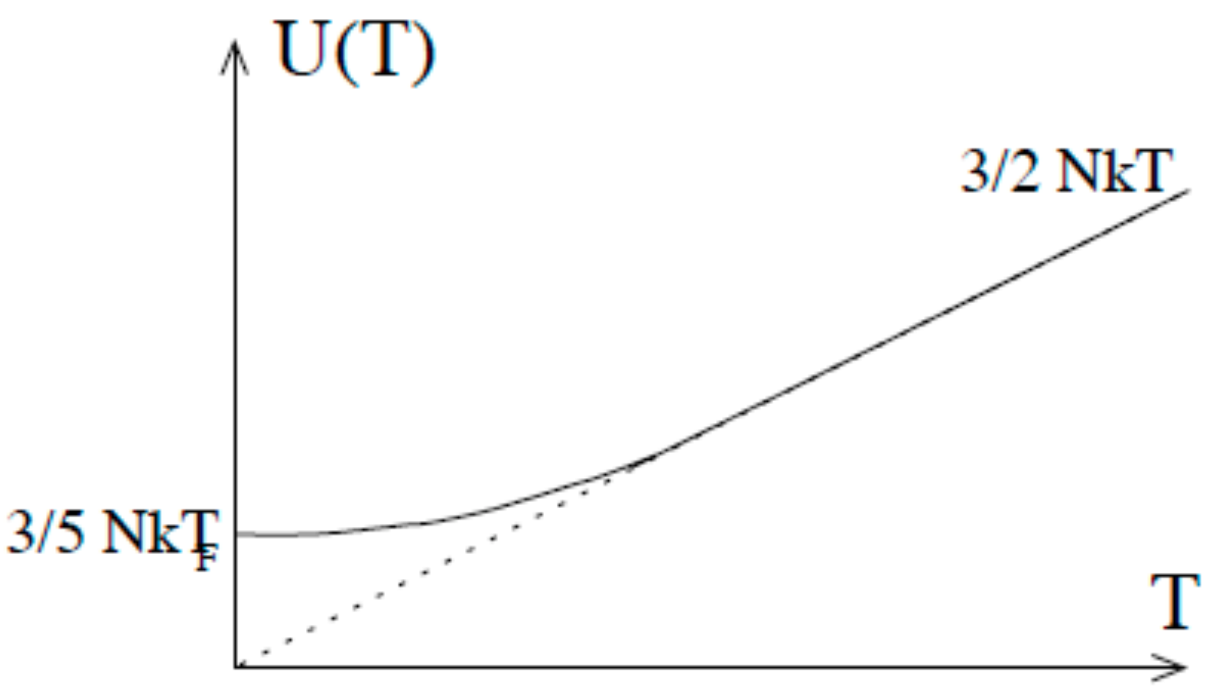
Ideales Fermi-Gas

$$U = -\frac{3}{2}\Omega = \frac{3}{2}PV$$

$$P(T=0)V = \frac{2}{5}Nk_\mathrm{B}T_F$$

$$k_\mathrm{B}T \ll \epsilon_F \quad \Rightarrow \quad U \approx \frac{3}{5}Nk_\mathrm{B}T_F$$

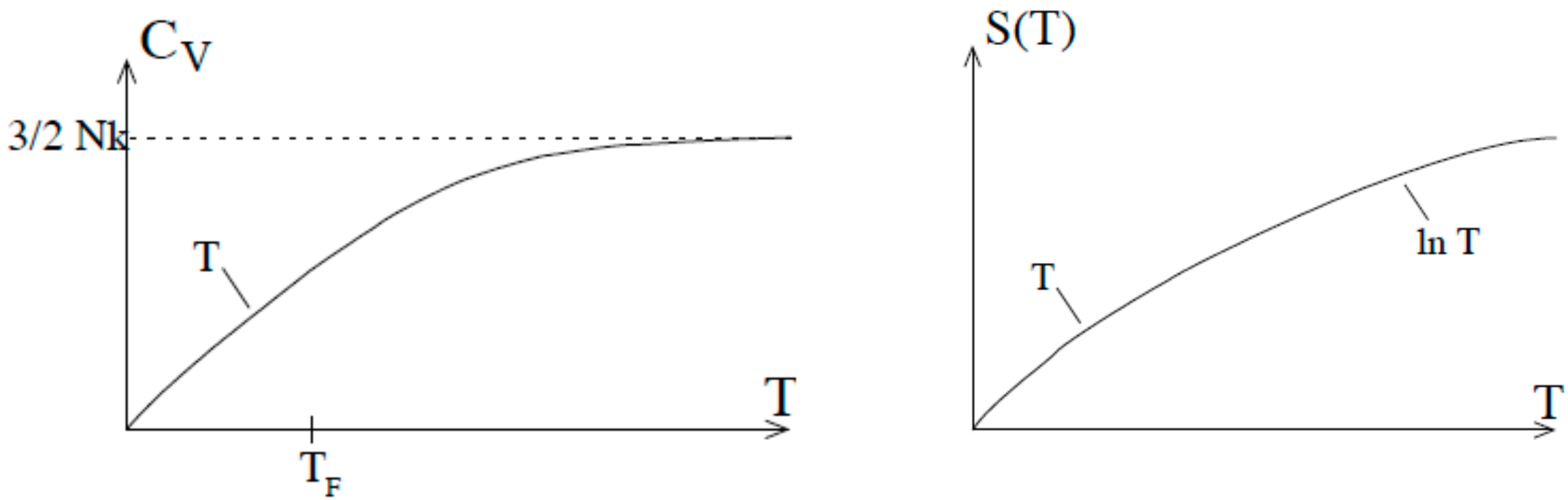
$$k_\mathrm{B}T \gg \epsilon_F \quad \Rightarrow \quad U = \frac{3}{2}k_\mathrm{B}TN$$



Ideales Fermi-Gas

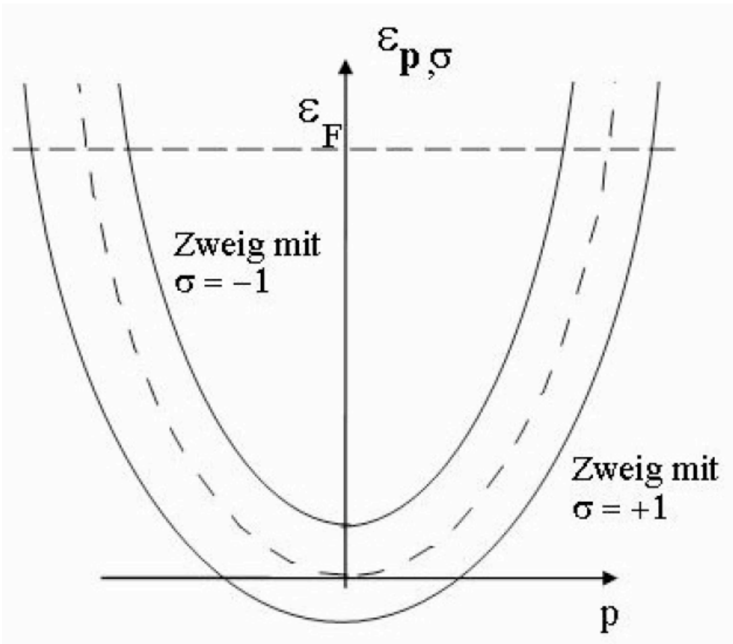
$k_\mathrm{B}T \ll \epsilon_F$ ⇒ $$C_V \approx N k_\mathrm{B}\,\frac{\pi^2}{2}\,\frac{k_\mathrm{B}T}{\epsilon_F}$$

$k_\mathrm{B}T \gg \epsilon_F$ ⇒ $$C_V \approx \frac{3}{2}\,N k_\mathrm{B}$$



Ideales Fermi-Gas

Pauli-Paramagnetismus



$$H = \frac{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}{2m} - \mu_B \sigma \mathbf{B} \qquad \mu_B = \frac{e\hbar}{2mc}$$

Diamagnetismus Paramagnetismus

$$\epsilon_{\mathbf{p},\sigma} = \frac{\mathbf{p}^2}{2m} - \mu_B \sigma B = \epsilon_{\mathbf{p}} - \mu_B \sigma B$$

$$Z_G = \sum_{n_{p,+}, n_{p,-}} e^{-\beta[(\epsilon_{\mathbf{p}} - \mu_B B - \mu) n_{\mathbf{p},+} + (\epsilon_{\mathbf{p}} + \mu_B B - \mu) n_{\mathbf{p},-}]}$$

$$\mu_\pm = \mu \pm \mu_B B$$

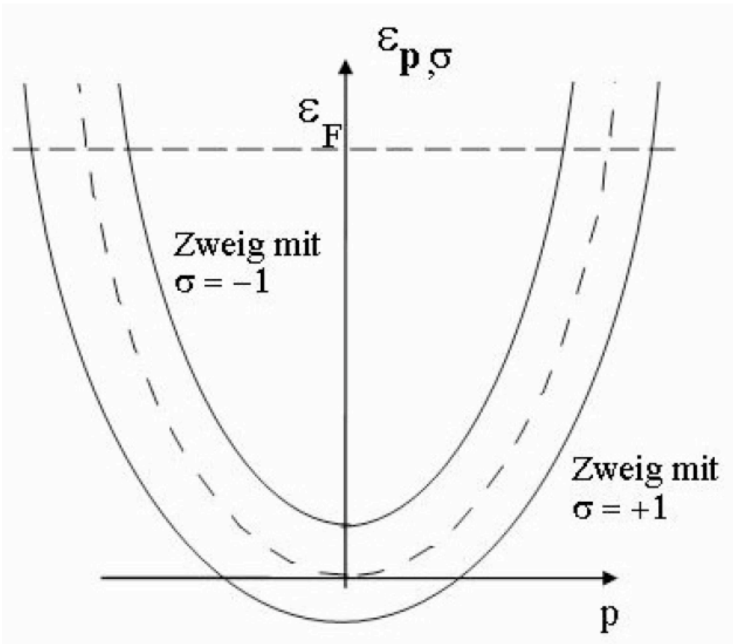
$$Z_G = Z_G^0(\mu_+) Z_G^0(\mu_-) \Rightarrow \Omega(T,V,\mu) = \Omega^0(T,V,\mu_+) + \Omega^0(T,V,\mu_-)$$

$$\Omega^0(T,V,\mu) = -k_{\mathrm{B}} T V \int d\epsilon\, \nu(\epsilon) \ln\left[1 + e^{-\beta(\epsilon - \mu)}\right] = -\frac{V}{\lambda_T^3} k_{\mathrm{B}} T f_{5/2}(z)$$

Ideales Fermi-Gas

Pauli-Paramagnetismus

$$\mu_\pm = \mu \pm \mu_B B$$



$$a(\epsilon) \equiv \int_{-\infty}^{\epsilon} d\epsilon_1 \nu(\epsilon_1) \qquad \nu(\epsilon) = \frac{m^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}$$

$$\frac{N_+ - N_-}{V} = a(\mu_+) - a(\mu_-) \approx 2\nu(\epsilon_F)\mu_B B$$

$k_\mathrm{B}T \ll \epsilon_F$

$$\frac{M}{V} = \mu_B \frac{(N_+ - N_-)}{V} = 2\nu(\epsilon_F)\mu_B^2 B = \chi B$$

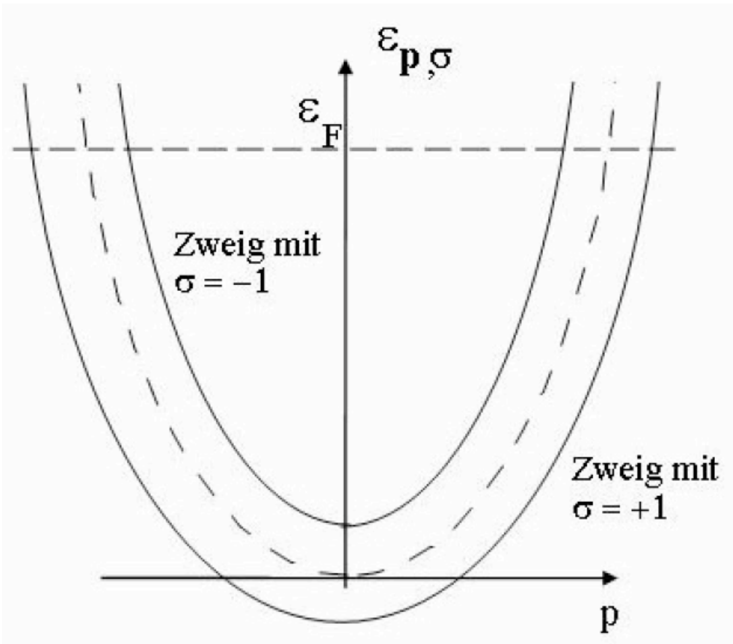
$k_\mathrm{B}T \gg \epsilon_F$

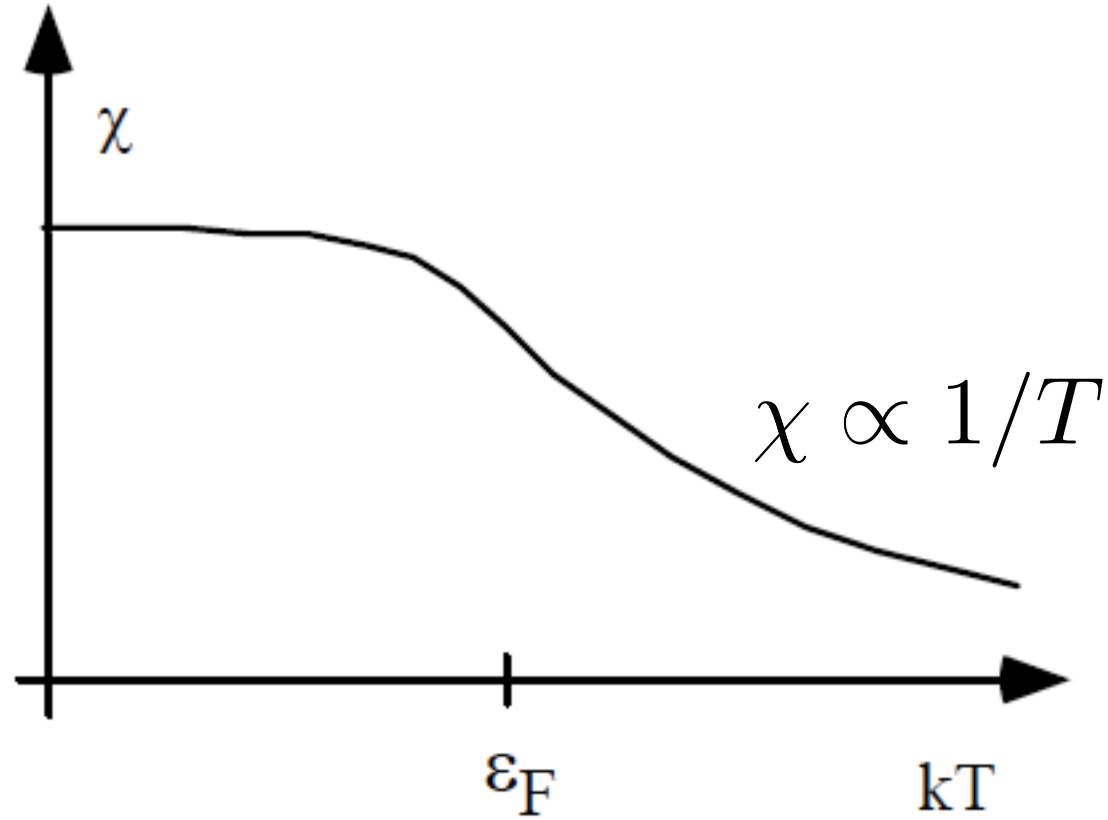
$$\begin{aligned}\frac{M}{V} &= \mu_B \frac{(N_+ - N_-)}{V} \approx \frac{\mu_\mathrm{B}}{\lambda_T^3}(e^{\beta\mu_+} - e^{\beta\mu_-}) \\ &= \frac{\mu_\mathrm{B}}{\lambda_T^3} e^{\beta\mu}\beta(2\mu_B B) = n\beta\mu_\mathrm{B}^2 B = \chi B\end{aligned}$$

$$N \approx (2s+1)\frac{V}{\lambda_T^3}\, z$$

Ideales Fermi-Gas

Pauli-Paramagnetismus





$k_\mathrm{B}T \ll \epsilon_F$ $\quad$ $\chi = 2\nu(\epsilon_F)\mu_B^2$ $\quad$ $\nu(\epsilon) = \dfrac{m^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}$

$k_\mathrm{B}T \gg \epsilon_F$ $\quad$ $\chi = n\beta\mu_\mathrm{B}^2 = \dfrac{n\mu_\mathrm{B}^2}{k_\mathrm{B}T}$