Karlsruher Institut für Technologie
Institut für Theorie der Kondensierten Materie

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1. Boltzmann equation in the presence of spin-orbit interaction: (100 Punkte)

In this exercise we practice solving the Boltzmann equation using the example of a system with spin-orbit interaction. We construct a model collision integral and attempt solving the resulting Boltzmann equation. As a result, we obtain the physics of the so-called Edelstein effect. Note, that this is not a true theory of the spin-orbit interaction, but rather an exercise: the collision integral we are using here is incorrect and as a result and although we do capture the main feature of the Edelstein effect – the spontaneous magnetization – the final expression for it that we obtain is wrong. It is however a good exercise that allows one to practice solving the kinetic equation. (A true kinetic theory of systems with spin-orbit interaction has not been worked out yet – it is an important topic in current research).

Consider a system with spin-orbit interaction as described by the Hamiltonian

$$H = \frac{p^2}{2m} + \boldsymbol{\Omega}(\boldsymbol{p}) \cdot \boldsymbol{\sigma},$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices and $\boldsymbol{\Omega}(\boldsymbol{p})$ is the fictitious "magnetic field" that depends on momentum and thus describes the spin-orbit interaction.

The Boltzmann equation is usually derived within the semiclassical approach, where we treat the quasiparticle momentum as a c-number (and not an operator). However, the electron spin has to be treated quantum-mechanically. This can be achieved by considering the one-particle density matrix which is a 2×2 matrix in spin space. The rest of the variables one can treat semiclassically, i.e. $\rho \to \rho_{\sigma_1\sigma_2}(\boldsymbol{r}, \boldsymbol{p}, t)$.

The kinetic equation can now be derived as usual. We consider time variation of the density matrix and equate it to the collision integral. The quantum-mechanical treatment of the spin variables amounts to using the well-known quantum-mechanical definition where the time derivative of an operator is given by its commutator with the Hamiltonian. This way in a spatially homogeneous systems one arrives at the equation

$$\frac{\partial \rho}{\partial t} + i\big[\boldsymbol{\Omega}(\boldsymbol{p}) \cdot \boldsymbol{\sigma}, \rho\big] - e\boldsymbol{E}\frac{\partial \rho}{\partial \boldsymbol{p}} = I[\rho].$$

Here $[\dots, \dots]$ stands for a commutator.

(a) Derive the above equation for a homogeneous system treating the spin variables quantum-mechanically and the momentum semiclassically. Simplify the equation for the steady state.

(b) Recall the well-known fact from quantum mechanics: any function of the Pauli matrices is a linear function. Therefore, the 2×2 density matrix can be written as

$$\rho = \frac{f}{2}\,\hat{1} + \boldsymbol{S} \cdot \boldsymbol{\sigma}.$$

Here $\hat{1}$ denotes a unity matrix.

Substitute this expression into the equation for the density matrix and find coupled equations for the charge and spin distribution fucntions f and $\boldsymbol{S}$.

(c) Consider the simplest version of the spn-orbit coupling in two-dimensional systems, the so-called Rashba spin-orbit coupling, which is described by

$$\boldsymbol{\Omega} = \alpha(p_y, -p_x).$$

Subsititute the Rashba form of the spin-orbit coupling in the equations obtained above.

Hint You are now dealing with a two-dimesnional system. The momentum is now a 2D vector, but spin still has three components.

(d) Now we will make the simplest assumptions allowing us to solve the resulting kinetic equations.

First, we assume that the external electric field is weak and the system is very close to its equilibrium state. Then the density matrix (as well as the distribution functions) can be written in the form

$$\rho = \rho^{(0)} + \delta\rho.$$

Assuming that the equilibrium state is described by the Fermi-Dirac distribution, find the explicit form of $\rho^{(0)}$, as well as $f^{(0)}$ and $S_j^{(0)}$.

(e) Finally, we assume that the collision integral can be treated in the τ-approximation. The resulting kinetic equations can be solved to the leading order in the applied electric field similarly to the standard derivation of the Drude formula given in the lecture.

Find the leading-order expression for the non-equilibrium corrections to the spin distribution functions δS_x and δS_y.

(f) Use the obtained distribution functions to find the average spin polarization in the system (the so-called Edelstein effect).