Lecture Notes, Field Theory in Condensed Matter Physics: Quantum Criticality and the Renormalization Group

Jörg Schmalian
Institute for Theory of Condensed Matter (TKM)
Karlsruhe Institute of Technology

Summer Semester, 2013

Contents

1 Introduction **1**

2 Classical models and classical critical phenomena **3**
2.1 The one-dimensional classical Ising model 3
2.1.1 Exact solution using transfer matrices 4
2.1.2 block-spin renormalization group 5
2.2 classical ϕ^4-theory 8
2.2.1 Lower and upper critical dimensions 8
2.2.2 Landau's mean field theory 10
2.3 Scaling laws 16
2.3.1 Fast and slow variables 20
2.4 ε-expansion of the ϕ^4-theory 22
2.4.1 Irrelevant interactions 27

3 The transverse field Ising-model **29**
3.1 Correspondence between quantum and classical theories 29
3.2 ε-expansion near $d = 3$ 29

4 Dilute bosons and fermions **31**
4.1 spin-dimers and dillute hard core bosons 31
4.2 renormalization group analysis for dilute bosons 31
4.3 dilute Fermions and Fermi liquid theory 31

5 Hertz-Millis theory of itinerant magnets **33**
5.1 motivation and RG-analysis 33
5.2 breakdown below the uppr critical dimension 33
5.3 crossover behavior above the upper critical dimension 33

6 Renormalization Group and Fermi liquid theory **35**
6.1 power-counting near a Fermi surface 35
6.2 one loop RG and fermionic instabilities 35
6.3 singular interactions and non-Fermi liquid fixed points 35

7 Quantum criticality in Graphene **37**
7.1 scale invariance in graphene 37
7.2 one and two loop RG near weak coupling 37
7.3 crossover analysis and observables 37

8 Disordered bosons **39**
8.1 quantum Harris criterion 39
8.2 perturbative renormalization group 39
8.3 quantum Griffiths effects 39
8.4 quantum percolation and non-Ginzburg-Landau behavior 39

Preface

These lecture notes summarize the main content of the course Field Theory in Condensed Matter Physics: Quantum Criticality and the Renormalization Group taught at the Karlsruhe Institute of Technology during the summer semester 2013.

Chapter 1

Introduction

To make quantitative predictions for interacting many body systems is notoriously difficult. The absence of an obvious small parameter that allows for a systematic theoretical investigation and the competition of a multitude of soft degrees of freedoms are the main obstacles. The behavior in the vicinity of a critical point, characterized by universal long wave length and low frequency behavior, is an important exception. The method of choice is the renormalization group. This lecture attempts to give a collection of problems that can be studied using renormalization group methods.

Chapter 2

Classical models and classical critical phenomena

2.1 The one-dimensional classical Ising model

In order to sharpen our language and to prepare important concepts we will briefly discuss a simple and solvable model of classical statistical mechanics, the one-dimensional Ising model. It is a nontrivial model of interacting spins in an external field, with Hamiltonian

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - \mu B \sum_i \sigma_i^z$$

Let $S_i = \pm 1$ be the eigenvalues of the Pauli matrix σ_i^z. Since all operators commute with each other and with the Hamiltonian we obtain immediately the many body eigenstates:

$$E\left(\{S_i\}\right) = -J \sum_i S_i S_{i+1} - \frac{\mu B}{2} \sum_i \left(S_i + S_{i+1}\right). \tag{2.1}$$

Although this model is obviously a quantum mechanical one, the fact that we have no problem determining the entire spectrum implies that it is often referred to as a classical problem.

The equilibrium statistical mechanics is governed by the partition function

$$\begin{aligned} Z &= \sum_{\{S_i\}} e^{-\beta E(\{S_i\})} \\ &= \sum_{S_1=\pm 1} \ldots \sum_{S_N=\pm 1} e^{\beta \sum_i \left[J S_i S_{i+1} + \frac{\mu B}{2} S_i + S_{i+1}\right]} \end{aligned} \tag{2.2}$$

2.1.1 Exact solution using transfer matrices

We use the method of transfer matrices and define the operator T defined via its matrix elements:

$$\langle S_i \left| T \right| S_{i+1} \rangle = e^{\beta \sum_i \left[J S_i S_{i+1} + \frac{\mu_B}{2}(S_i + S_{i+1}) \right]} \tag{2.3}$$

The operator can be represented as 2×2 matrix

$$T = \begin{pmatrix} e^{\beta(J+\mu_B B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-\mu_B B)} \end{pmatrix} \tag{2.4}$$

It holds then that

$$\begin{aligned} Z &= \sum_{S_1=\pm 1} ... \sum_{S_N=\pm 1} \langle S_1 \left| T \right| S_2 \rangle \langle S_2 \left| T \right| S_3 \rangle ... \langle S_N \left| T \right| S_1 \rangle \\ &= \sum_{S_1=\pm 1} \left\langle S_1 \left| T^N \right| S_1 \right\rangle = \mathrm{tr} T^N. \end{aligned} \tag{2.5}$$

This can be expressed in terms of the eigenvalues of the matrix T

$$\lambda_\pm = e^K \cosh b \pm \left[e^{-2K} + e^{2K} \sinh^2 h \right]^{1/2} \tag{2.6}$$

with

$$\begin{aligned} h &= \beta \mu_{\mathrm{B}} B \\ K &= \beta J. \end{aligned} \tag{2.7}$$

It follows

$$Z = \lambda_+^N + \lambda_-^N \tag{2.8}$$

yielding

$$\begin{aligned} F &= -k_{\mathrm{B}} T \left(N \log \lambda_+ + \log \left(1 + \left(\frac{\lambda_-}{\lambda_+} \right)^N \right) \right) \\ &= -k_{\mathrm{B}} T N \log \lambda_+ \end{aligned} \tag{2.9}$$

where we used in the last step that $\lambda_- < \lambda_+$ and took the thermodynamic limit $N \to \infty$. For the non-interacting system ($J = 0$) we obtain immediately the result of free spins

$$F = -N k_{\mathrm{B}} T \log \left(2 \cosh \frac{\mu_{\mathrm{B}} B}{k_B T} \right). \tag{2.10}$$

For arbitrary J follows for the magnetization:

$$M = -\frac{\partial F}{\partial B} = N \mu_B \frac{\sinh \frac{\mu_{\mathrm{B}} B}{k_B T}}{\left(e^{-\frac{4J}{k_B T}} + \sinh^2 \frac{\mu_{\mathrm{B}} B}{k_B T} \right)^{1/2}}. \tag{2.11}$$

It holds $M(B \to 0, T = 0) = \pm N\mu_B$, i.e. the system is fully polarized along the infinitesimal field direction. On the other hand it holds for any finite temperature that

$$M(T, B \to 0) \to 0. \tag{2.12}$$

Thus, there is no ordered state with finite zero-field magnetization at finite temperature. The one dimensional Ising model orders only at zero temperature.

2.1.2 block-spin renormalization group

Next we perform an analysis very similar to the renormalization group. In performing the trace over the spin configurations we will sum over every second spin site. For example we perform the trace over S_2

$$\begin{aligned} f_{S_1,S_3} &= \sum_{S_2=\pm 1} e^{K(S_1S_2+S_2S_3)+h(S_1+S_2+S_3)} \\ &= e^{K(S_1+S_3)+h(S_1+1+S_3)} + e^{-K(S_1+S_3)+h(S_1-1+S_3)}. \end{aligned} \tag{2.13}$$

Now, this expression only depends on S_1 and S_3, the two neighboring spins of S_2. This can easily be written in a more compact fashion. It holds that

$$f_{S_1,S_2} = \exp\left(2g + K'S_1S_2 + h'(S_1 + S_2)\right), \tag{2.14}$$

with

$$\begin{aligned} \log f_{++} &= 2g + 2h' + K' \\ \log f_{+-} &= 2g - K' \\ \log f_{--} &= 2g - 2h' + K', \end{aligned} \tag{2.15}$$

which leads to

$$\begin{aligned} K' &= \frac{1}{2}\log\left(\frac{\sqrt{f_{++}f_{--}}}{f_{+-}}\right) \\ h' &= \frac{1}{4}\log\frac{f_{++}}{f_{--}} \\ g &= \frac{1}{4}\log\left(\sqrt{f_{++}f_{--}}f_{+-}\right) \end{aligned}$$

Lets consider first the case without magnetic field, $b = 0$.

$$\begin{aligned} K' &= \frac{1}{4}\log\left(\frac{\cosh(2K+h)\cos(2K-h)}{\cosh^2 h}\right) \\ h' &= h + \frac{1}{2}\log\left(\frac{\cosh(2K+h)}{\cosh(2K-h)}\right) \\ g &= \frac{1}{8}\log\left(16\cosh(2K+h)\cosh(2K-h)\cosh^2 h\right) \end{aligned} \tag{2.16}$$

Thus, up to a constant, the partition sum over all sites can be written as the partition sum over every second site with identical form of the Hamiltonian

$$\begin{aligned} \beta H &= K\sum_{i=1}^{N}\sigma_i^z\sigma_{i+1}^z - h\sum_i \sigma_i^z \longrightarrow \\ \beta H_{\text{eff}} &= K'\sum_{i \text{ even},\ i=1}^{N}\sigma_i^z\sigma_{i+2}^z - h'\sum_{i \text{ even},\ i=1}^{N}\sigma_i^z \end{aligned} \tag{2.17}$$

Thus, we obtain the following scaling relation:

$$Z\left(N,K,h\right) = e^{Ng(K,b)}Z\left(N/2,K',h'\right). \tag{2.18}$$

The constant g will not affect the calculation of any expectation value, but it will enter into a calculation of the total free energy. It represents the contribution to the free energy from short wavelength degrees of freedom which have been traced out.

We can now repeat this procedure n-times and trace out each time every second spin. Eq.2.16 are the recursion relations of this procedure. In each step, the number spins is reduced by half, i.e.

$$N' = N/b \quad \text{with } b = 2^n. \tag{2.19}$$

Alternatively the lattice spacing between neighboring sites increases according to $a' = ba$. If we measure length scales in terms of the underlying lattice constants it must hold

$$\xi' = \xi/b, \tag{2.20}$$

where for example ξ is the physical correlation length of the system of interest and ξ' is the correlation length of the system with renormalized values for K' and b' that occur after n-iterations. For the free energy follows

$$F\left(N,K,h\right) = F\left(N/2,K',h'\right) - Nk_BTg\left(K,h\right) \tag{2.21}$$

which implies for the free energy density $f = F/N$ after multiple iterations

$$f\left(K,h\right) = b^{-1}f\left(K',h'\right) - k_BTg\left(K,h\right) \tag{2.22}$$

At fist glance, mapping a Hamiltonian onto itself doesn't seem to make life easier or harder, it only seems to be a waste of time. However, from the way the parameters of the Hamiltonian flow under repeated renormalization, we can learn a lot about the long distance physics. For example, if we are in a regime where

$$K' = K, \text{ and } b' = b \tag{2.23}$$

we know that it doesn't matter whether we eliminate degrees of freedom or not. We get the same behavior on all length scales. For obvious reasons parameters where Eq.2.23 is fulfilled correspond to a fixed point of the Hamiltonian.

Let us analyze the recursion relations. We introduce

$$x = e^{-4K} \text{ and } y = e^{-2h} \tag{2.24}$$

and obtain

$$\begin{aligned} x' &= x\frac{(1+y)^2}{(x+y)(1+xy)} \\ y' &= y\frac{x+y}{1+xy} \end{aligned} \tag{2.25}$$

Consider first $h = 0$, i.e. $y = 1$, which yields $y' = 1$, i.e. one fixed point corresponds to $B = 0$. Then

$$x' = \frac{4x}{(1+x)^2} \geq x \tag{2.26}$$

Thus, with some initial value $0 < x < 1$ it holds that $x' > x$. Thus x grows under the flow approaching the fixed point $x^* = 1$. As the flow is towards this value, it is referred to as the stable fixed point. Of course $x^* = 1$ corresponds to $K^* = 0$. For $T > 0$, the coupling between spins gets weaker and weaker for longer and longer distances. This is consistent with the statement that there is no magnetic order above $T = 0$. For $T = 0$, it holds $x = 0$ and we have another fixed point $x^* = 0$, i.e. $K^* = \infty$. This is the zero-temperature or strong coupling fixed point, where all spins are strongly coupled on arbitrarily large distances. It corresponds to the above determined long range magnetic order. Since the flow goes away from $x^* = 0$ for arbitrarily small but finite x, this fixed point is unstable.

At low temperatures holds $x' \simeq 4x$, which implies

$$x(n) = 4^n x \tag{2.27}$$

or

$$K(n) = K - \frac{1}{2}n\log 2 \tag{2.28}$$

Roughly after about $n_0 \simeq 2K/\log 2$ iterations holds that $K(n_0)$ is negligibly small. Then it should hold that renormalized correlation length is of order of one lattice constant (of the decimated lattice). Thus, we have from $\xi' = \xi/b$ with $\xi' \simeq 1$ if $b = 2^{n_0}$, yielding

$$\xi \simeq 2^{n_0} = e^{n_0 \log 2} = e^{2J/T}. \tag{2.29}$$

Thus, we find that the correlation length is finite for any finite temperature and diverges exponentially as $T \to 0$. The exact result for the correlation length, defined via

$$\left\langle \sigma_i^z \sigma_j^z \right\rangle \propto \exp\left(-|i-j|/\xi\right) \tag{2.30}$$

is

$$\xi = -\frac{1}{\log(\tanh(K))} \tag{2.31}$$

in the limit of large $K = \beta J$ we obtain $\xi = 2e^{2J/T}$ in full agreement with our above estimate. The determination of the correlation length is an example for

2.2 classical ϕ^4-theory

The classical ϕ^4-theory is determined by the effective Hamilonian

$$H[\phi] = \frac{1}{2}\int d^d r \left(r_0 \phi^2 + (\nabla\phi)^2 + \frac{u}{2}\phi^4 \right). \tag{2.32}$$

and determines the partition function

$$Z = \int D\phi \exp\left(-\beta H[\phi]\right). \tag{2.33}$$

Here $D\phi = \prod_k d\phi_k$ where ϕ_k are the Fourier modes of the field ϕ_k. If the underlying microscopic theory is defined on a lattice with lattice constant a, no Fourier coefficients with $k > 2\pi/a$ should occur. In addition, the above effective Hamiltonian should be understood as a coarse grained version of the theory, i.e. it should be valid only for wavelengths that are longer that the typical microscopic length scale. This gives rise to an upper momentum cut off Λ that is smaller that $2\pi/a$. Modes are suppressed for $k > \Lambda$.

During the tutorials you will discuss how to derive this coarse grained continuum's theory from the lattice Ising model. In addition, there is no reason to confine ourselves to ϕ being a real number. For example, in case of continuously varying spins, as described by the XY- or Heisenberg models, the order parameter is an N-component vector $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$, with $N = 2$ or 3, respectively. It is often interesting and convenient to consider N-component order parameters with arbitrary N. Then the ϕ^4-model becomes

$$H[\phi] = \frac{1}{2}\int d^d r \left(r_0 \phi\cdot\phi + \nabla\phi\cdot\nabla\phi + \frac{u}{2}(\phi\cdot\phi)^2 \right) \tag{2.34}$$

where $\phi\cdot\phi = \sum_{i=1}^N \phi_i^2$.

2.2.1 Lower and upper critical dimensions

The role of fluctuations depends sensitively on the dimension d. As we will see, the space-dimension where fluctuations become important is referred to as the upper critical dimension $d_{\rm uc}$. In case of the ϕ^4-theory we will find that $d_{\rm uc} = 4$. You will discuss during the tutorial that $d_{\rm uc} = 6$ for a ϕ^3-model. The notion of an upper critical dimension makes only sense if there is in fact long range order and a finite transition temperature. For example, in case of the one-dimensional Ising model we found that no finite transition temperature exists. Whether long range order occurs or not can we estimated rather easily and determines the lower critical dimension $d_{\rm lc}$. We will now show that $d_{\rm lc} = 1$ for systems with discrete order parameter symmetry (and short range interactions), while $d_{\rm lc} = 2$ for systems with continuous order parameter.

Let us first consider a discrete order parameter, like the Ising model. Suppose we have an ordered state with a given order parameter $\phi(x) = \phi_0$. Now lets

estimate the free energy cost of a defect of the perfectly ordered state. In case of the Ising model (i.e. scalar ϕ^4-model) such a defect could be a droplet of size L. Suppose inside the droplet holds $\phi(x) = -\phi_0$, while outside of it the order parameter has the opposite sign. A typical configuration would be

$$\phi(r) \simeq \phi_0 \tanh\left(\frac{r-L}{l_0}\right) \tag{2.35}$$

where l_0 is some microscopic length scale. It follows for the energy of the defect

$$E_{defect} = \int d^d x \left(\nabla\phi\right)^2 \simeq \phi_0^2 \frac{1}{l_0} L^{d-1} \tag{2.36}$$

This could have been equally estimated in terms of the original Ising model where $E_{defect} \simeq JL^{d-1}$. The typical energy of such a defect is

$$F_{defect} \simeq \phi_0^2 \frac{1}{l_0} L^{d-1} - T \log L^d, \tag{2.37}$$

where the second term is the entropy associated with the possible arrangement of the droplet. Such droplets will certainly occur and they are the natural mechanism that reduce then order parameter to a smaller but finite value at $0 < T < T_c$. As long as $d > 1$, the interface-energy will always dominate over the entropy-gain of the defect, making large droplets increasingly unlikely. However, for $d = 1$ holds that the interface energy is independent of size (the surface consists of only two points no matter how large the droplet size L). There is nothing to prevent arbitrarily large droplets and droplets in other droplets etc. to occur. Thus, long range order will not occur. This is of course in complete agreement with our exact result for the one-dimensional Ising model. It holds $d_{\mathrm{lc}} = 1$ as mentioned above.

Next we consider a continuously varying order parameter, i.e. $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$. In comparison to the scalar order parameter, we can now simply rotate the order parameter vector without changing its amplitude. For example

$$\phi = \phi_0 \left(\cos\theta(x), \sin\theta(x), \cdots, 0\right) \tag{2.38}$$

with

$$\theta(x) = \begin{cases} \pi\left(1 - \frac{|x|}{L}\right) & |x| < L \\ 0 & |x| \geq L \end{cases} \tag{2.39}$$

Then, the order parameter outside the defect is $\phi(|x| \geq L) = (1, 0, \cdots, 0)$ and it slowly rotates to reach $\phi(x = 0) = (0, 1, \cdots, 0)$. Since $\phi \cdot \phi = \phi_0^2$ the nonlinear ϕ^4 term is completely unaffected by this defect, in distinction to the single component order parameter that had to "climb over the hill" of the energy landscape and locally melt to zero. The penalty comes from

$$E_{defect} = \int d^d x \left(\nabla\phi\right)^2 = \phi_0^2 \int d^d x \left(\nabla\theta\right)^2 \simeq \phi_0^2 L^{d-2}, \tag{2.40}$$

yielding

$$F_{defect} \simeq \phi_0^2 L^{d-2} - T \log L^d. \tag{2.41}$$

The same reasoning as before leads to $d_{\rm lc} = 2$ for systems with continuously varying order parameter. No long range order is possible in $d = 2$ and below. This statement can be proven rigorously and goes back to Hohenberg as well as Mermin and Wagner.

2.2.2 Landau's mean field theory

A first attempt to solve these problems is to approximate the integral by the dominant contribution of the integrand, i.e. we write

$$\int D\phi \exp\left(-\beta H\left[\phi\right]\right) \simeq \exp\left(-\beta H\left[\phi_0\right]\right) \tag{2.42}$$

where $\frac{\delta H}{\delta \phi}\Big|_{\phi=\phi_0} = 0$. This leads to the Landau theory of phase transitions. Of course, in general, it is not only the minimum of $H\left[\phi\right]$ w.r.t. ϕ which corresponds to physically realized configurations. Instead one has to integrate over all values of ϕ to obtain the free energy. Before we do this, we briefly discuss the Landau theory.

Landau proposed that one should introduce an order parameter to describe the properties close to a phase transition. This order parameter should vanish in the high temperature phase and be finite in the ordered low temperature phase. The mathematical structure of the order parameter depends strongly on the system under consideration. In case of an Ising model the order parameter is a scalar, in case of the Heisenberg model it is a vector. For example, in case of a superconductor or the normal fluid - superfluid transition of ^{4}He it is a complex scalar, characterizing the wave function of the coherent low temperature state. Another example are nematic liquid crystals where the order parameter is a second rank tensor.

In what follows we will first develop a Landau theory for a scalar, Ising type order. Landau argued that one can expand the free energy density in a Taylor series with respect to the order parameter ϕ. This should be true close to a second order transition where ϕ vanishes continuously:

$$\mathcal{H}\left(\phi\right) = -h\phi + \frac{r_0}{2}\phi^2 + \frac{b}{3}\phi^3 + \frac{u}{4}\phi^4 + ... \tag{2.43}$$

The physical value of the order parameter is the determined by minimizing $\mathcal{H}\left(\phi\right)$

$$\left.\frac{\partial \mathcal{H}\left(\phi\right)}{\partial \phi}\right|_{\phi=\phi_0} = 0. \tag{2.44}$$

If $u < 0$ this minimum will be at $\pm\infty$ which is unphysical. If indeed $u < 0$ one needs to take a term $\sim \phi^6$ into account and see what happens. In what follows we will always assume $u > 0$. In the absence of an external field should hold

that $\mathcal{H}(\phi) = \mathcal{H}(-\phi)$, implying $h = b = 0$. Whether or not there is a minimum for $\phi \neq 0$ depends now on the sign of r_0. If $r_0 > 0$ the only minimum of

$$\mathcal{H}(\phi) = f_0 + \frac{r_0}{2}\phi + \frac{u}{4}\phi^4 \tag{2.45}$$

is at $\phi = 0$. However, for $r_0 < 0$ there are two a new solutions $\phi = \pm\sqrt{\frac{-r_0}{u}}$. Since ϕ is expected to vanish at $T = T_c$ we conclude that $r_0(T)$ changes sign at T_c suggesting the simple ansatz

$$r_0(T) = a_0(T - T_c) \tag{2.46}$$

with $a_0 > 0$ being at most weakly temperature dependent. This leads to a temperature dependence of the order parameter"

$$\phi_0 = \begin{cases} \sqrt{\frac{a_0(T_c - T)}{u}} & T < T_c \\ 0 & T > T_c \end{cases} . \tag{2.47}$$

It will turn out that a powerlaw relation like

$$\phi \sim (T_c - T)^{\beta} \tag{2.48}$$

is valid in a much more general context. The main change is the value of β. The prediction of the Landau theory is $\beta = \frac{1}{2}$.

Next we want to study the effect of an external field (= magnetic field in case ϕ characterizes the magnetization of an Ising ferromagnet). This is done by keeping the term $h\phi$ in the expansion for f. The actual external field will be proportional to h. Then we find that f is minimized by

$$r_0\phi_0 + u\phi_0^3 = h \tag{2.49}$$

Right at the transition temperature where $a = 0$ this gives

$$\phi_0 \sim h^{1/\delta} \tag{2.50}$$

where the Landau theory predicts $\delta = 3$. Finally we can analyze the change of the order parameter with respect to an external field. We introduce the susceptibility

$$\chi = \left.\frac{\partial\phi_0}{\partial h}\right|_{h\to 0} \tag{2.51}$$

and find from Eq.2.49

$$r_0\chi + 3u\phi_0^2(h = 0)\chi = 1 \tag{2.52}$$

using the above result for $\phi_0^2(h = 0) = \frac{r_0}{u}$ if $T < T_c$ and $\phi_0^2(h = 0) = 0$ above T_c gives

$$\chi = \begin{cases} \frac{1}{4a_0}(T_c - T)^{-\gamma} & T < T_c \\ \frac{1}{a_0}(T - T_c)^{-\gamma} & T > T_c \end{cases} \tag{2.53}$$

with exponent $\gamma = 1$.

Next we consider the specific heat where we insert our solution for ϕ_0 into the free energy density.

$$\mathcal{H}(\phi_0) = \frac{r_0}{2}\phi_0^2 + \frac{u}{4}\phi_0^4 = \begin{cases} -\frac{a_0^2}{4u}(T - T_c)^2 & T < T_c \\ 0 & T > T_c \end{cases} \tag{2.54}$$

This yields for the specific heat per volume

$$C = -T\frac{\partial^2 f}{\partial T} = -T\frac{\partial^2 \mathcal{H}(\phi_0)}{\partial T} = \begin{cases} \frac{a_0^2}{4u}T & T < T_c \\ 0 & T > T_c \end{cases} . \tag{2.55}$$

The specific heat is discontinuous. As we will see later, the general form of the specific heat close to a second order phase transition is

$$C(T) \sim (T - T_c)^{-\alpha} + const \tag{2.56}$$

where the result of the Landau theory is

$$\alpha = 0. \tag{2.57}$$

In our analysis of the Landau theory we only considered spatially homogeneous solutions of the order parameter. Next, we include the more general case of spatially varying order parameters, i.e.

$$H = \int d^d r \mathcal{H}[\phi] \tag{2.58}$$

is given as

$$\mathcal{H}[\phi] = \frac{r_0}{2}\phi(r)^2 + \frac{u}{4}\phi(r)^4 - h(r)\phi(r) + \frac{1}{2}(\nabla\phi(r))^2 \tag{2.59}$$

where we assumed that it costs energy to induce an inhomogeneity of the order parameter. In addition we assumed that we can always absorb the coefficient $\frac{1}{2}(\nabla\phi(r))^2$ into the definition of the order parameter. The variational minimum $\frac{\delta H}{\delta\phi}\Big|_{\phi=\phi_0} = 0$ of H is now determined by the Euler-Lagrange equation

$$\frac{\partial \mathcal{H}}{\partial \phi} - \nabla\frac{\partial \mathcal{H}}{\partial \nabla\phi} = 0 \tag{2.60}$$

which leads to the nonlinear partial differential equation

$$r_0\phi(r) + u\phi(r)^3 = h(r) + \nabla^2\phi(r). \tag{2.61}$$

Above the transition temperature we neglect again the non-linear term and have to solve

$$r_0\phi(r) - \nabla^2\phi(r) = h(r) \tag{2.62}$$

It is useful to consider the generalized susceptibility

$$\delta\phi(r) = \int d^d r' \chi(r - r') \delta h(r') \tag{2.63}$$

which determines how much a local change in the order parameter is affected by a local change of an external field at a distance $r - r'$. This is often written as

$$\chi(r - r') = \frac{\delta\phi(r)}{\delta h(r')}. \tag{2.64}$$

We determine $\chi(r - r')$ by Fourier transforming the above differential equation with

$$\phi(r) = \int \frac{d^d k}{(2\pi)^d} e^{-ikr} \phi(k) \tag{2.65}$$

which gives

$$r_0 \phi(k) + k^2 \phi(k) = h(k) \tag{2.66}$$

In addition it holds for $\chi(k)$:

$$\delta\phi(k) = \chi(k) \delta h(k). \tag{2.67}$$

This leads to

$$\chi(k) = \frac{1}{\xi^{-2} + k^2} \tag{2.68}$$

where we introduced the length scale

$$\xi = \sqrt{\frac{1}{r_0}} = \sqrt{\frac{1}{a_0}} (T - T_c)^{-1/2} \tag{2.69}$$

This result can now be back-transformed yielding at large distances

$$\chi(r - r') = \left(\frac{\xi}{|r - r'|}\right)^{\frac{d-1}{2}} \exp\left(-\frac{|r - r'|}{\xi}\right) \tag{2.70}$$

Thus, spins are not correlated anymore beyond the correlation length ξ. In general the behavior of ξ close to T_c can be written as

$$\xi \sim (T - T_c)^{-\nu} \tag{2.71}$$

with $\nu = \frac{1}{2}$. A similar analysis can be performed in the ordered state. Starting again at

$$r_0 \phi(r) + u\phi(r)^3 = h(r) + \nabla^2 \phi(r) \tag{2.72}$$

and assuming $\phi(r) = \phi_0 + \psi(r)$ where ϕ_0 is the homogeneous, $h = 0$, solution, it follows for small $\psi(r)$:

$$\left(r_0 + 3u\phi_0^2\right) \psi(r) = h(r) + \nabla^2 \psi(r) \tag{2.73}$$

and it holds $r_0 + 3u\phi_0^2 = -2r_0 > 0$. Thus in momentum space

$$\chi(k) = \frac{d\psi(k)}{dh(k)} = \frac{1}{\xi_<^{-2} + k^2} \tag{2.74}$$

with

$$\xi = \sqrt{\frac{1}{-2r_0}} = \sqrt{\frac{1}{2a_0}} (T_c - T)^{-1/2} \tag{2.75}$$

Ginzburg criterion

One can now estimate the range of applicability of the Landau theory. This is best done by considering the next order corrections and analyze when they are small. If this is the case, one can be confident that the theory is controlled. Before we go into this we need to be able to perform some simple calculations with these multidimensional integrals.

First we consider for simplicity a case where $H_{\text{eff}}[\phi]$ has only quadratic contributions. It holds

$$\begin{aligned} Z &= \int D\phi \exp\left(-\frac{1}{2}\sum_k \phi_k \left(r_0 + \mathbf{k}^2\right)\phi_{-k}\right) \\ &= \prod_k \int d\phi_k \exp\left(-\frac{1}{2}\phi_k \left(r_0 + \mathbf{k}^2\right)\phi_{-k}\right) \\ &= \prod_k \left(\frac{(2\pi)^d}{r_0 + \mathbf{k}^2}\right)^{1/2} = \exp\left(\frac{1}{2}\sum_k \log\chi(k)\right) \end{aligned}$$

with

$$\chi(k) = \frac{1}{r_0 + \mathbf{k}^2}. \tag{2.76}$$

It follows for the free energy

$$F = -\frac{k_B T}{2} N \int \frac{d^d k}{(2\pi)^d} \log\chi(k) \tag{2.77}$$

One can also add to the Hamiltonian an external field

$$H[\phi] \to H[\phi] - \int d^d k h(k)\phi(k) \tag{2.78}$$

Then it is easy to determine the correlation function

$$\chi(k) = \langle\phi_k\phi_{-k}\rangle - \langle\phi_k\rangle\langle\phi_{-k}\rangle \tag{2.79}$$

via

$$\begin{aligned} \left.\frac{\delta \log Z}{\delta h_k \delta h_{-k}}\right|_{h\to 0} &= \frac{\delta}{\delta h_k}\frac{1}{Z}\int D\phi\phi_k e^{-\beta H_{\text{eff}}[\phi]} \\ &= \frac{1}{Z}\int D\phi\phi_k\phi_{-k} e^{-\beta H_{\text{eff}}[\phi]} - \frac{\left(\int D\phi\phi_k e^{-\beta H_{\text{eff}}[\phi]}\right)^2}{Z^2} \\ &= \chi(k) \end{aligned} \tag{2.80}$$

This can again be done explicitly for the case with $u = 0$:

$$\begin{aligned} Z[h] &= \int D\phi \exp\left(-\frac{1}{2}\int d^d k \phi_k \left(a + b\mathbf{k}^2\right)\phi_{-k} + \int d^d k h(k)\phi_k\right) \\ &= Z[0]\exp\left(\frac{1}{2}\int d^d k h_k \chi(k) h_{-k}\right) \end{aligned} \tag{2.81}$$

Performing the second derivative of $\log Z$ gives indeed $\left\langle \phi_k \phi_{-k} \right\rangle = \frac{1}{r_0 + \mathbf{k}^2}$. Thus, we obtain as expected

$$\chi(k) = \frac{\delta \phi_k}{\delta h_{-k}}. \tag{2.82}$$

Let us analyze the specific heat related to the free energy

$$F = -\frac{k_B T}{2} N \int d^d k \log \chi(k) \tag{2.83}$$

It holds for the singular part of the specific heat

$$C \sim -\frac{\partial^2 F}{\partial r_0^2} \sim \int d^d k \chi(k)^2 \sim \int \frac{k^{d-1} dk}{\left(\xi^{-2} + k^2\right)^2} \sim \xi^{4-d} \tag{2.84}$$

Thus, as $\xi \to \infty$ follows that there is no singular (divergent) contribution to the specific heat if $d > 4$ just as we found in the Landau theory. However, for $d < 4$ the specific heat diverges and we obtain a behavior different from what Landau theory predicted.

Another way to see this is to study the role of inhomogeneous fluctuations as caused by the

$$H_{inh} = \frac{1}{2}\int d^d r (\nabla \phi)^2 \tag{2.85}$$

Consider a typical variation on the scale $\nabla \phi \sim \sqrt{\frac{-r_0}{u}}\xi^{-1}$ and integrate those over a volume of size ξ^d gives

$$H_{inh} \sim \xi^{d-2}\frac{r_0}{u} \sim \frac{1}{u}\xi^{d-4} \tag{2.86}$$

Those fluctuations should be small compared to temperature in order to keep mean field theory valid. If their energy is large compared to $k_B T$ they will be rare and mean field theory is valid. Thus we obtain again that mean field theory breaks down for $d < 4$. This is called the Ginzburg criterion. Explicitly this criterion is

$$\xi^{-1} > (u k_B T)^{\frac{1}{4-d}}. \tag{2.87}$$

Note, if b is large for some reason, fluctuation physics will enter only very close to the transition. This is indeed the case for many so called conventional superconductors.

2.3 Scaling laws

A crucial observation of our earlier results of second order phase transitions was the divergence of the correlation length

$$\xi\left(T \to T_c\right) \to \infty. \tag{2.88}$$

This divergency implies that at the critical point no characteristic length scale exists, which is in fact an important reason for the emergence of the various power laws. Using h as a dimensionless number proportional to an external field and

$$r = \frac{T - T_c}{T_c} \tag{2.89}$$

as dimensionless measure of the distance to the critical point the various critical exponents are:

$$\begin{aligned} \xi\left(r, h=0\right) &\sim r^{-\nu} \\ \phi\left(r, h=0\right) &\sim |r|^{\beta} \\ \phi\left(r=0, h\right) &\sim h^{1/\delta} \\ \chi\left(r, h=0\right) &\sim r^{-\gamma} \\ C\left(r, h=0\right) &\sim r^{-\alpha} \\ \chi\left(x \to \infty, r=0\right) &\sim x^{2-d-\eta}. \end{aligned} \tag{2.90}$$

where D is the spatial dimensionality. The values of the critical exponents for a number of systems are given in the following table

exponent	mean field	$d=2$, Ising	$d=3$, Ising
α	0	0	0.12
β	$\frac{1}{2}$	$\frac{1}{8}$	0.31
γ	1	$\frac{7}{4}$	1.25
ν	$\frac{1}{2}$	1	0.64
δ	3	15	5.0
η	0	$\frac{1}{4}$	0.04

It turn out that a few very general assumptions about the *scaling* behavior of the correlation function $\chi\left(q\right)$ and the free energy are sufficient to derive very general relations between these various exponents. Those relations are called *scaling laws*. We will argue that the fact that there is no typical length scale characterizing the behavior close to a second order phase transition leads to a powerlaw behavior of the singular contributions to the free energy and correlation function. For example, consider the result obtained within Landau theory

$$\chi\left(q, r\right) = \frac{1}{r + q^2} \tag{2.91}$$

where we eliminated irrelevant prefactors. Rescaling all length r of the system according to $x \to x/b$, where b is an arbitrary dimensionless number, leads to

$k \to kb$. Obviously, the mean field correlation function obeys

$$\chi(q,r) = b^2 \chi\left(bq, b^2 r\right). \tag{2.92}$$

Thus, upon rescaling ($k \to kb$), the system is characterized by a correlation function which is the same up to a prefactor and a readjustment of the distance from the critical point. In what follows we will generalize this expression and assume that even beyond the mean field theory of Landau a similar relationship holds

$$\chi(q,r) = b^{2-\eta} \chi\left(bq, b^y r\right). \tag{2.93}$$

The mean field theory is obviously recovered if $y = 2$ and $\eta = 0$. Since b is arbitrary, we can for example chose $tb^y = 1$ implying $b = t^{-\frac{1}{y}}$ and we obtain directly from our above ansatz

$$\chi(q,t) = r^{-\frac{2-\eta}{y}} \chi\left(qr^{-\frac{1}{y}}, 1\right). \tag{2.94}$$

By definition, the correlation length is the length scale which characterizes the momentum variation of $\chi(q,r)$ i.e. $\chi(q,r) \sim f(q\xi)$, which leads to $\xi \sim r^{-\frac{1}{y}}$ and we obtain

$$\nu = y^{-1}. \tag{2.95}$$

The exponent y of our above ansatz for $\chi(q,r)$ is therefore directly related to the correlation length exponent. This makes it obvious why it was necessary to generalize the mean field behavior. $y = 2$ yields the mean field value of ν. Next we consider $r = 0$ and chose $bq = 1$ such that

$$\chi(q, r=0) = \frac{1}{q^{2-\eta}} \chi(1,0) \tag{2.96}$$

which gives

$$\chi(x, r=0) = \int \frac{d^d q}{(2\pi)^d} \chi(q, r=0)\, e^{ikx} \sim \int dq e^{ikx} \frac{q^{d-1}}{q^{2-\eta}} \tag{2.97}$$

substituting $z = kx$ gives

$$\chi(x, r=0) \sim x^{2-d-\eta}. \tag{2.98}$$

Thus, the exponent η of Eq.2.93 is indeed the same exponent as the one given above. This exponent is often called anomalous dimension and characterizes the change in the powerlaw decay of correlations at the critical point (and more generally for length scales smaller than ξ). Thus we can write

$$\chi(q,r) = b^{2-\eta} \chi\left(bq, b^{\frac{1}{\nu}} r\right). \tag{2.99}$$

Similar to the correlation function can we also make an assumption for the free energy density

$$f(r,h) = b^{-d} F\left(rb^y, hb^{y_h}\right). \tag{2.100}$$

The prefactor b^{-d} is a simple consequence of the fact that an extensive quantity changes upon rescaling of length with a corresponding volume factor. Using $y = \nu^{-1}$ we can again use $tb^y = 1$ and obtain

$$f(r,h) = r^{d\nu} F\left(1, hr^{-\nu y_h}\right). \tag{2.101}$$

This enables us to analyze the specific heat at $h = 0$ as

$$C \sim \frac{\partial^2 F(r,0)}{\partial r^2} \sim r^{d\nu - 2} \tag{2.102}$$

which leads to

$$\alpha = 2 - d\nu. \tag{2.103}$$

This is a highly nontrivial relationship between the spatial dimensionality, the correlation length exponent and the specific heat exponent. It is our first scaling law. Interestingly, it is fulfilled in mean field (with $\alpha = 0$ and $\nu = \frac{1}{2}$) only for $d = 4$.

The temperature variation of the order parameter is given as

$$\phi(r) \sim \left.\frac{\partial f(r,h)}{\partial h}\right|_{h\to 0} \sim r^{\nu(d-y_h)} \tag{2.104}$$

which gives

$$\beta = \nu(d - y_h) = 2 - \alpha - \nu y_h \tag{2.105}$$

This relationship makes a relation between y_h and the critical exponents just like y was related to the exponent ν. Within mean field

$$y_h = 3 \tag{2.106}$$

Alternatively we can chose $hb^{y_h} = 1$ and obtain

$$f(r,h) = h^{\frac{d}{y_h}} f\left(rh^{-\frac{1}{\nu y_h}}, 0\right) \tag{2.107}$$

This gives for the order parameter at the critical point

$$\phi(r = 0, h) \sim \frac{\partial f(r = 0, h)}{\partial h} \sim h^{\frac{d}{y_h} - 1} \tag{2.108}$$

and gives $\frac{1}{\delta} = \frac{d}{y_h} - 1$. One can simplify this to

$$\delta = \frac{y_h}{d - y_h} = \frac{2 - \alpha - \beta}{\beta} \tag{2.109}$$

and yields

$$\beta(1 + \delta) = 2 - \alpha \tag{2.110}$$

Note, the mean field theory obeys $\delta = \frac{y_h}{y_h - d}$ only for $d = 4$. whereas $\delta = \frac{2-\alpha-\beta}{\beta}$ is obeyed by the mean field exponents for all dimensions. This is valid quite generally, scaling laws where the dimension, d, occurs explicitly are fulfilled

within mean field only for $d = 4$ whereas scaling laws where the dimensionality does not occur are valid more generally.

The last result allows us to rewrite our original ansatz for the free energy

$$f(r,h) = b^{(2-\alpha)\nu^{-1}} f\left(rb^{\frac{1}{\nu}}, hb^{\frac{\beta\delta}{\nu}}\right). \tag{2.111}$$

such that $tb^{\frac{1}{\nu}} = 1$ leads to

$$f(r,h) = f^{2-\alpha} r\left(1, ht^{-\beta\delta}\right) \tag{2.112}$$

We next analyze how the susceptibility diverges at the critical point. It holds

$$\chi \sim \left.\frac{\partial^2 f(r,h)}{\partial h^2}\right|_{h\to 0} \sim r^{2-\alpha-2\beta\delta} \tag{2.113}$$

which leads to

$$\gamma = \alpha - 2 + 2\beta\delta \tag{2.114}$$

which is yet another scaling relation.

The last scaling law follows from the fact that the correlation function $\chi(q,r)$ taken at $q = 0$ equals the susceptibility χ just analyzed. This gives

$$\chi(r) = b^{2-\eta}\chi(rb^y) \tag{2.115}$$

and choosing again $rb^y = 1$ yields

$$\chi(r) = r^{-\nu(2-\eta)} \tag{2.116}$$

such that

$$\gamma = \nu(2-\eta). \tag{2.117}$$

To summarize, we have identified all the exponents in the assumed scaling relations of $F(t,h)$ and $\chi(q,t)$ with critical exponents (see Eqn.2.99 and2.111). In addition we have four relationships the six exponents have to fulfill at the same time which are collected here:

$$\begin{aligned} \alpha &= 2 - d\nu. \\ \beta(1+\delta) &= 2 - \alpha \\ 2\beta\delta - \gamma &= 2 - \alpha \\ \gamma &= \nu(2-\eta) \end{aligned} \tag{2.118}$$

One can easily check that the exponents of the two and three dimensional Ising model given above indeed fulfill all these scaling laws. If one wants to calculate these exponents, it turns out that one only needs to determine two of them, all others follow from scaling laws.

2.3.1 Fast and slow variables

The divergency which caused the break down of Landau theory was caused by long wave length, i.e. the $k \to 0$ behavior of the integral which renormalized $u \to u'$. One suspicion could be that only long wave length are important for an understanding of this problem. However, this is not consistent with the scaling concept, where the rescaling parameter was always assumed to be arbitrary. In fact it fluctuations on *all* length scales are crucial close to a critical point. This is on the one hand a complication, on the other hand one can take advantage of this beautiful property. Consider for example the scaling properties of the correlation function

$$\chi(q,r) = b^{2-\eta}\chi\left(bq, rb^{\frac{1}{\nu}}\right). \tag{2.119}$$

Repeatedly we chose $rb^{\frac{1}{\nu}} = 1$ such that $b = r^{-\nu} \to \infty$ as one approaches the critical point. However, if this scaling property (and the corresponding scaling relation for the free energy) are correct for generic b (of course only if the system is close to T_c) one might analyze a rescaling for b very close to 1 and infer the exponents form this more "innocent" regime. If we obtain a scaling property of $\chi(q,r)$ it simply doesn't matter how we determined the various exponents like ν, η etc.

This, there are two key ingredients of the renormalization group. The first is the assumption that scaling is a sensible approach, the second is a decimation procedure which makes the scaling transformation $x \to x/b$ explicit for $b \simeq 1$. A convenient way to do this is by considering $b = e^l$ for small l. Lets consider a field variable

$$\phi(\mathbf{k}) = \int d^d x \exp(i\mathbf{k}\cdot\mathbf{x})\,\phi(\mathbf{x}) \tag{2.120}$$

Since there is an underlying smallest length-scale $\overline{a}$ ($\simeq$interatomic spacing), no waves with wave number larger than a given upper cut off $\Lambda \simeq \overline{a}^{-1}$ should occur. For our current analysis the precise value of Λ will be irrelevant, what matters is that such a cut off exists. Thus, be observe that $\phi(k) = 0$ if $k > \Lambda$.

We need to develop a scheme which allows us to explicitly rescale length or momentum variables. How to do this goes back to the work of Leo Kadanoff and Kenneth G. Wilson in the early 70^{th} of the last century. The idea is to divide the typical length variations of $\phi(k)$ into short and long wave length components

$$\phi(k) = \begin{cases} \phi^{<}(k) & 0 < k \le \Lambda/b \\ \phi^{>}(k) & \Lambda/b < k \le \Lambda \end{cases}. \tag{2.121}$$

If one now eliminates the degrees of freedoms $\phi^{>}$ one obtains a theory for $\phi^{<}$ only

$$\exp\left(-H'\left[\phi^{<}\right]\right) = \int D\phi^{>} \exp\left(-H\left[\phi^{<},\phi^{>}\right]\right). \tag{2.122}$$

The momenta in $H'\left[\phi^{<}\right]$ are confined to the smaller region $0 < k \le \Lambda/b$. We can now rescale simply according to

$$k' = bk \tag{2.123}$$

such that the new variable k' is restricted to the original scales $0 < k' \leq \Lambda$. The field variable is then $\phi^{<}(k'/b)$ and will conveniently be called

$$\phi'(k') = b^{-\rho}\phi^{<}(k'/b) \tag{2.124}$$

where the prefactor $b^{-\rho}$ is only introduced for later convenience to be able to keep the prefactor of the k^2 term in the Hamiltonian the same. The *renormalized* Hamiltonian is then determined by $H'\left[\phi'\right]$.

In practice we start for example from a theory of the type Eq.2.34 and obtain a renormalized Hamiltonian

$$H(r,u) \rightarrow H'(r(l),u(l)). \tag{2.125}$$

If one now analyzes the so-called flow equation of the parameters $r(l)$, $u(l)$ etc. there are a number of distinct cases. The most interesting one occurs if one approaches a fixed point where $r(l \rightarrow \infty) = r^*$, $u(l \rightarrow \infty) = u^*$ etc. If this is the case the low energy behavior of the system is identical for all initial values which reach the fixed point.

Scaling behavior of the correlation function:

We start from $H[\phi]$ with cut off scale Λ. The new Hamiltonian with cut off Λ/b, which results from the shell integration, is then determined by

$$e^{-H'[\phi^{<}]} = \int D\phi^{>} e^{-H[\phi^{<},\phi^{>}]}, \tag{2.126}$$

which is supplemented by the rescaling

$$\phi^{<}(k) = b^{\rho}\phi'(bk)$$

which yields the new Hamiltonian $H'\left[\phi'\right]$ which is governed by the same cut off Λ. If one considers states with momenta with $k < \Lambda/b$, it is possible to determine the corresponding correlation function either from $H[\phi]$ or from $H'\left[\phi'\right]$. Thus, we can either start from the original action:

$$\langle\phi(k_1)\phi(k_2)\rangle = \int \frac{D\phi e^{-H[\phi]}}{Z}\phi(k_1)\phi(k_2) = \chi(k_1)\delta(k_1+k_2) \tag{2.127}$$

or, alternatively, use the renormalized action:

$$\begin{aligned}\langle\phi(k_1)\phi(k_2)\rangle &= \int \frac{D\phi' e^{-H'[\phi']}}{Z'} b^{2\rho}\phi'(bk_1)\phi'(bk_1) \\ &= b^{2\rho}\chi'(kb_1)\delta(bk_1+bk_2) \\ &= b^{2\rho-d}\chi'(bk_1)\delta(k_1+k_2)\end{aligned} \tag{2.128}$$

where $\chi'(bk) = \chi(bk, r(l), u(l))$ is the correlation function evaluated for H' i.e. with parameters $r(l)$ and $u(l)$ instead of the "bare" ones r and u, respectively. It follows

$$\chi(k,r,u) = b^{2\rho-d}\chi(k,r(l),u(l)) \tag{2.129}$$

This is close to an actual derivation of the above scaling assumption and suggests to identify

$$2\rho - d = 2 - \eta. \tag{2.130}$$

What is missing is to demonstrate that $r(l)$ and $u(l)$ give rise to a behavior $te^{yl} = tb^y$ of some quantity t which vanishes at the phase transition. To see this is easier if one performs the calculation explicitly.

2.4 ε-expansion of the ϕ^4-theory

We will now follow the recipe outlined in the previous paragraphs and explicitly calculate the functions $r(l)$ and $u(l)$. It turns out that this can be done in a controlled fashion for spatial dimensions close to $d = 4$ and we therefore perform an expansion in $\varepsilon = 4 - d$. First we consider the free part of the Hamiltonian given by:

$$H_0(\boldsymbol{\phi}) = \frac{1}{2}\int^{\Lambda} \frac{d^d k}{(2\pi)^d}\ \chi_0^{-1}(k)\ \boldsymbol{\phi}(k)\cdot\boldsymbol{\phi}(-k) \tag{2.131}$$

where

$$\chi_0^{-1}(k) = r_0 + k^2. \tag{2.132}$$

Incidentally, if we want to determine the correlation function, it follows from usual Gaussian integration:

$$\langle \phi(k)\,\phi(-k)\rangle = \frac{\int D\phi\phi(k)\,\phi(-k)\,e^{-H_0}}{\int D\phi e^{-H_0}} = \chi_0(k). \tag{2.133}$$

Here we used $\int d\phi\phi^2 \exp\left(-\frac{1}{2\chi_0}\phi^2\right) = \sqrt{2\pi}\chi_0^{3/2}$ and $\int d\phi \exp\left(-\frac{1}{2\chi_0}\phi^2\right) = \sqrt{2\pi}\chi_0^{1/2}$.

Integrating out states in the momentum shell between Λ/b and Λ with $b = e^l$ yields an additive correction δF to the free energy and we are left with the effective action of states $\boldsymbol{\phi}^<(q)$ with momenta smaller than Λ/b:

$$H_0'(\boldsymbol{\phi}^<) = \frac{1}{2}\int^{\Lambda/b} \frac{d^d k}{(2\pi)^d}\ \left(r + k^2\right)\ \boldsymbol{\phi}^<(k)\cdot\boldsymbol{\phi}^<(-k) \tag{2.134}$$

In order to recover the original form of the action we finally rescale momentum and temperature via: $\mathbf{k}' = b\mathbf{k}$ and $\boldsymbol{\phi}'(k') = Z_\phi^{1/2}\boldsymbol{\phi}^<(k)$ with $Z_\phi = b^{-2\rho}$. Choosing $2\rho = d + 2$ gives finally the renormalized action:

$$H_0'(\boldsymbol{\phi}') = \frac{1}{2}\int^{\Lambda} \frac{d^d k'}{(2\pi)^d}\ \left(r(l) + k'^2\right)\ \boldsymbol{\phi}'(k')\cdot\boldsymbol{\phi}'(-k') \tag{2.135}$$

The renormalized parameter $r(l)$ obeys the following equation:

$$\frac{dr(l)}{dl} = 2r(l)$$

Next we consider the quartic term

$$H_{\rm int} = \frac{u}{4}\int d^dk_1 d^dk_2 d^dk_3 \phi\left(k_1\right)\phi\left(k_2\right)\phi\left(k_3\right)\phi\left(-k_1-k_2-k_3\right) \tag{2.136}$$

which couples $\phi^>$ and $\phi^<$. If all three momenta are inside the inner shell, we can easily perform the rescaling and find

$$H'_{\rm int} = \frac{ub^{4\rho-3d}}{4}\int d^Dk'_1 d^Dk'_2 d^Dk'_3 \phi\left(k'_1\right)\phi\left(k'_2\right)\phi\left(k'_3\right)\phi\left(-k'_1-k'_2-k'_3\right) \tag{2.137}$$

which gives with the above result for ρ:

$$4\rho - 3d = 4 - d \tag{2.138}$$

yielding

$$u\left(l\right) = ue^{(4-d)l}. \tag{2.139}$$

which is equivalent:

$$\frac{du(l)}{dl} = (4-d)u(l)\,. \tag{2.140}$$

Thus, at tree level, the Gaussian fixed point $u^* = 0$ is unstable if $d < 4$. If however some of the momenta in $H_{\rm int}$ are in the outer shell and others have Fourier modes inside Λ/b we need to analyze the coupling between them. Integrating out of states in the momentum shell between Λ/b and Λ can be performed using

$$\begin{aligned}\exp\left(-H'(\boldsymbol{\phi}^<)\right) &\propto \exp\left(-H'_0(\boldsymbol{\phi}^<)\right)\left\langle\exp\left(-H_{int}(\boldsymbol{\phi}^<,\boldsymbol{\phi}^>)\right)\right\rangle_> \\ &\equiv \exp\left(-H'_0(\boldsymbol{\phi}^<) - \delta H'(\boldsymbol{\phi}^<)\right)\end{aligned} \tag{2.141}$$

where the average $\langle\cdots\rangle_>$ is with respect the spin excitations, $\boldsymbol{\phi}^>$, with momenta between Λ/b and Λ. Within the one loop approximation, it is useful to use the cumulant expansion:

$$\delta H' = \langle H_{int}\rangle_> - \frac{1}{2}\left(\left\langle H^2_{int}\right\rangle_> - \langle H_{int}\rangle^2_>\right) + \cdots\,. \tag{2.142}$$

Explicitly it holds for the correction term to the action:

$$\begin{aligned}H_{int}(\boldsymbol{\phi}^<,\boldsymbol{\phi}^>) &= \frac{u}{4}\int^\Lambda \frac{d^dk_1}{(2\pi)^d}\cdots\int^\Lambda \frac{d^dk_4}{(2\pi)^d}(\mathbf{S}^<(k_1)+\mathbf{S}^>(k_1))\cdot(\mathbf{S}^<(k_2)+\mathbf{S}^>(k_2)) \\ &\quad (\mathbf{S}^<(k_3)+\mathbf{S}^>(k_3))\cdot(\mathbf{S}^<(k_4)+\mathbf{S}^>(k_4))\;\delta_{k_1+k_2+k_3+k_4}\,.\end{aligned} \tag{2.143}$$

Averaging with respect to the $\boldsymbol{\phi}^>(k)$ excitations yields non-vanishing contributions only if the number of $\boldsymbol{\phi}^>(k)$ fields in the corresponding term is even. The case with zero $\boldsymbol{\phi}^>(k)$ was already discussed above. It is referred to as the tree level contribution to the interaction term. The term with four $\boldsymbol{\phi}^>(k)$ fields yields a constant which renormalizes the free energy on the two loop level. Finally, there are in case of $N = 3$ altogether 10 contributions with two $\boldsymbol{\phi}^>(k)$

fields. Here the remaining two field carry momentum $|\mathbf{k}| < \Lambda/b$. These terms renormalize the free Hamiltonian. In the case of an $O(N)$ symmetric vector field, there are altogether $2(N+2)$ contributions of this kind, yielding

$$\begin{aligned}\delta S_0'(\mathbf{S}^<) &= 4(N+2)\frac{u}{4}\int^> \frac{d^dq}{(2\pi)^d}\left\langle\phi(q)\,\phi(-q)\right\rangle_> \frac{1}{2}\int^{\Lambda/b}\frac{d^dk}{(2\pi)^d}\,\boldsymbol{\phi}^<(k)\cdot\boldsymbol{\phi}^<(-k)\\ &= (N+2)u\int^>\frac{d^dq}{(2\pi)^d}G(p)\,\frac{1}{2}\int^{\Lambda/b}\frac{d^dk}{(2\pi)^d}\,\boldsymbol{\phi}^<(k)\cdot\boldsymbol{\phi}^<(-k)\,, \qquad (2.144)\end{aligned}$$

where $\int^> d^dq$ denotes a momentum integration with $|\mathbf{q}|$ between Λ/b and Λ.

Considering the second term on the right hand side of Eq 2.142, which is of order u^2, at the one loop level it is sufficient to consider renormalizations of the interaction part of the action. All renormalizations of the free part will contain two closed loops and are beyond the present single loop approximation. There are altogether $8(N+8)$ combinations to contract spin fields $\boldsymbol{\phi}^<(k)$ leaving four $\boldsymbol{\phi}^>(k)$ fields which can finally be expressed as:

$$\begin{aligned}\delta H_{int}'(\boldsymbol{\phi}^<) &= -\frac{1}{2}8(N+8)\left(\frac{u}{4}\right)^2\int^{\Lambda/b}\frac{dk_1}{(2\pi)^d}\cdots\int^{\Lambda/b}\frac{dk_4}{(2\pi)^d}\,\boldsymbol{\phi}^<(k_1)\cdot\boldsymbol{\phi}^<(k_2)\boldsymbol{\phi}^<(k_3)\cdot\boldsymbol{\phi}^<(k_4)\\ &\quad\times\int^>\frac{d^dp}{(2\pi)^d}G(p)G(q_1+q_2-p)\;\delta_{k_1+k_2+k_3+k_4} \qquad (2.145)\end{aligned}$$

From these considerations we can finally obtain the renormalization group equations for the correlation length and coupling constant within the one loop approximation, which replace Eq. **??** and 2.140:

$$\begin{aligned}r' &= e^{2l}r + (N+2)u\int^>\frac{d^dq}{(2\pi)^d}G(p)\\ u' &= e^{(4-d)l}u - (N+8)u^2\int^>\frac{d^dq}{(2\pi)^d}G(p)G(-p). \qquad (2.146)\end{aligned}$$

The key difference to a straightforward perturbation theory is, that the momentum integration is restricted to the shell with radius between Λ/b and Λ. This avoids all the complications of a direct perturbation theory where a divergency in u' would result from the lower limit of the integration (long wave lengths). Integrals of the type

$$I = \int_{\Lambda/b<k<\Lambda}\frac{d^dk}{(2\pi)^d}f(k) = K_d\int_{\Lambda e^{-l}}^{\Lambda}k^{d-1}f(k)\,dk \qquad (2.147)$$

where the integration over angles yields $K_d = \frac{2}{\left(2\sqrt{\pi}\right)^d\Gamma(d/2)}$, i.e. $K_2 = \frac{1}{2\pi}$, $K_3 = \frac{1}{2\pi^2}$, or $K_4 = \frac{1}{8\pi^2}$. For small l follows:

$$I \simeq K_d\Lambda^{d-1}f(\Lambda)\left(\Lambda - \Lambda e^{-l}\right) \simeq K_d\Lambda^d f(\Lambda)\,l$$

It holds therefore

$$\begin{aligned} r' &= (1+2l)\, r + \frac{(N+2)\, K_d \Lambda^d}{r+\Lambda^2} u l \\ u' &= (1+\varepsilon l)\, u - \frac{(N+8)\, K_d \Lambda^d}{(r+\Lambda^2)^2} u^2 l \ , \end{aligned} \tag{2.148}$$

which is due to the small-l limit conveniently written as a differential equation

$$\begin{aligned} \frac{dr}{dl} &= 2r + \frac{(N+2)\, K_d \Lambda^d}{r+\Lambda^2} u, \\ \frac{du}{dl} &= \varepsilon u - \frac{(N+8 K_d \Lambda^d}{(r+\Lambda^2)^2} u^2. \end{aligned} \tag{2.149}$$

Before we proceed, we introduce more convenient variables

$$\begin{aligned} r &\to \frac{r}{\Lambda^2} \\ u &\to K_d \Lambda^{d-4} u \end{aligned} \tag{2.150}$$

which are dimensionless and obtain the differential equations

$$\begin{aligned} \frac{dr}{dl} &= 2r + \frac{(N+2)\, u}{1+r} \\ \frac{du}{dl} &= \varepsilon u - \frac{(N+8) u^2}{(1+r)^2}. \end{aligned} \tag{2.151}$$

The system has indeed a fixed point (where $\frac{dr}{dl} = \frac{du}{dl} = 0$) determined by

$$\begin{aligned} \varepsilon &= \frac{(N+8)\, u^*}{(1+r^*)^2} \\ 2r^* &= -\frac{(N+2)\, u^*}{1+r^*} \end{aligned} \tag{2.152}$$

This simplifies at leading order in ε to

$$\begin{aligned} u^* &= \frac{\varepsilon}{N+8} \text{ or } 0 \\ r^* &= -\frac{(N+2)}{2} u^* \end{aligned} \tag{2.153}$$

If the system reaches this fixed point it will be governed by the behavior in its immediate vicinity, allowing us to linearize the flow equation in the vicinity of the fixed point, i.e. for small

$$\begin{aligned} \delta r &= r - r^* \\ \delta u &= u - u^* \end{aligned} \tag{2.154}$$

Consider first the fixed point with $u^* = r^* = 0$ gives

$$\frac{d}{dl}\begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & \varepsilon \end{pmatrix}\begin{pmatrix} \delta r \\ \delta u \end{pmatrix} \tag{2.155}$$

with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \varepsilon$. Both eigenvalues are positive for $\varepsilon > 0$ ($D < 4$) such that there is no scenario under which this fixed point is ever governing the low energy physics of the problem.

Next we consider $u^* = \frac{\varepsilon}{9}$ and $r^* = -\frac{\varepsilon}{6}$. It follows

$$\frac{d}{dl}\begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = \begin{pmatrix} 2 - \frac{N+2}{N+8}\varepsilon & N + 2 + \frac{(N+2)^2}{2(N+8)}\varepsilon \\ 0 & -\varepsilon \end{pmatrix}\begin{pmatrix} \delta r \\ \delta u \end{pmatrix} \tag{2.156}$$

with eigenvalues

$$\begin{aligned} y &= 2 - \frac{N+2}{N+8}\varepsilon \\ y' &= -\varepsilon \end{aligned} \tag{2.157}$$

the corresponding eigenvectors are

$$\begin{aligned} e &= (1, 0) \\ e' &= \left(-\frac{3\,(N+2)\,(N+6)}{4\,(N+8)} + \frac{\varepsilon}{8}, 1\right) \end{aligned} \tag{2.158}$$

Thus, a variation along the e-direction (which is varying r) causes the system to leave the fixed point (positive eigenvalue), whereas it will approach the fixed point if

$$(r, u) \sim e' \tag{2.159}$$

this gives

$$r_c\,(u) = u\left(-\frac{3\,(N+2)\,(N+6)}{4\,(N+8)} + \frac{\varepsilon}{8}\right) \tag{2.160}$$

which defines the *critical surface* in parameter space. If a system is on this surface it approaches the fixed point. If it is slightly away, the quantity

$$t = r - r_c\,(u) \tag{2.161}$$

is non-zero and behaves as

$$t\,(l) = te^{yl} = tb^y. \tag{2.162}$$

The flow behavior for large l is only determined by the value of t which is the only scaling variable, which vanishes at the critical point. Returning now to the initial scaling behavior of the correlation function we can write explicitly

$$\chi\,(k, t) = b^2\chi\,(k, tb^y) \tag{2.163}$$

comparing this with $\chi(q,t) = b^{2-\eta}\chi\left(bq, tb^{\frac{1}{\nu}}\right)$ gives immediately the two critical exponents

$$\begin{aligned} \eta &= \mathcal{O}\left(\varepsilon^2\right) \\ \nu &\simeq \frac{1}{2} + \frac{N+2}{N+8}\frac{\varepsilon}{4}. \end{aligned} \tag{2.164}$$

A systematic improvement of these results occurs if one includes higher order terms of the εexpansion. Thus, the renormalization group approach is a very powerful tool to analyze the highly singular perturbation expansion of the ϕ^4-theory below its upper critical dimension. How is it possible that one can obtain so much information by essentially performing a low order expansion in u for a small set of high energy degrees of freedom? The answer is in the power of the scaling concept. We have assumed that the form $\chi(q,t) = b^{2-\eta}\chi\left(bq, tb^{\frac{1}{\nu}}\right)$ which we obtained for very small deviations of b from unity is valid for all b. If for example the value of ν and η would change with l there would be no way that we could determine the critical exponents from such a procedure. If scaling does not apply, no critical exponent can be deduced from the renormalization group.

2.4.1 Irrelevant interactions

Finally we should ask why we restricted ourself to the quartic interaction only. For example, one might have included a term of the type

$$H_{(6)} = \frac{v}{6}\int d^d x \phi(x)^6 \tag{2.165}$$

which gives in momentum space

$$\begin{aligned} H_{(6)} &= \frac{v}{6}\int d^d k_1 ... d^d k_5 \phi(k_1)\phi(k_2)\phi(k_3)(k_4)\phi(k_5) \\ &\quad \times\phi(-k_1-k_2-k_3-k_4-k_5) \end{aligned} \tag{2.166}$$

The leading term of the renormalization group is the one where all three momenta are inside the inner shell, and we can perform the rescaling immediately:

$$\begin{aligned} H'_{\text{int}} &= \frac{vb^{6\rho-5d}}{5}\int d^d k'_1 ... d^d k'_5 \phi(k'_1)\phi(k'_2)\phi(k'_3)\phi(k'_4)\phi(k'_5) \\ &\quad \times\phi(-k'_1-k'_2-k'_3-k'_4-k'_5) \end{aligned} \tag{2.167}$$

and the v dependence is with $\rho = \frac{2+d}{2}$ and $6\rho - 5d = 2(3-d) = -2(1-\varepsilon)$

$$v(l) = ve^{-2(1-\varepsilon)l} \tag{2.168}$$

Thus, in the strict sense of the ε expansion such a term will never play a role. Only if $u \ll \varepsilon$ initially is it important to keep these effects into account. This happens in the vicinity of a so called tricritical point.

Chapter 3

The transverse field Ising-model

3.1 Correspondence between quantum and classical theories

3.2 ε-expansion near $d = 3$

Chapter 4

Dilute bosons and fermions

4.1 spin-dimers and dillute hard core bosons

4.2 renormalization group analysis for dilute bosons

4.3 dilute Fermions and Fermi liquid theory

Chapter 5

Hertz-Millis theory of itinerant magnets

5.1 motivation and RG-analysis

5.2 breakdown below the uppr critical dimension

5.3 crossover behavior above the upper critical dimension

Chapter 6

Renormalization Group and Fermi liquid theory

6.1 power-counting near a Fermi surface

6.2 one loop RG and fermionic instabilities

6.3 singular interactions and non-Fermi liquid fixed points

Chapter 7

Quantum criticality in Graphene

7.1 scale invariance in graphene

7.2 one and two loop RG near weak coupling

7.3 crossover analysis and observables

Chapter 8

Disordered bosons

8.1 quantum Harris criterion

8.2 perturbative renormalization group

8.3 quantum Griffiths effects

8.4 quantum percolation and non-Ginzburg-Landau behavior