KARLSRUHER INSTITUT FÜR TECHNOLOGIE

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Feldtheorie der Kondensierten Materie, RG und Quantenkritikalität SS 2013

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1. Ising ϕ^4-field theory (10 + 20 + 10 + 10 + 10 + 10 = 70 Punkte)

In this exercise we derive the scalar ϕ^4-field theory starting from the microscopic Ising Hamiltonian

$$H = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij} S_i S_j \,, \tag{1}$$

where i, j run over lattice sites (*e.g.* of a d-dimensional cubic lattice), the Ising variables take values $S_i = \pm 1$, $J_{ij} > 0$ denotes a ferromagnetic coupling between spins.

(a) Prove the following identity for multi-dimensional integrals over real variables

$$\int_{-\infty}^{\infty} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2}} e^{-\frac{1}{2}\sum_{i,j} x_i A_{ij} x_j + \sum_i x_i J_i} = [\det A]^{-1/2} e^{\frac{1}{2}\sum_{ij} J_i (A^{-1})_{ij} J_j} \,,$$

where A is a real symmetric positive definite matrix.

(b) Use this identity to make the exponent of the partition sum for the Ising model linear in the spin variables S_i (Hubbard-Stratonovich transformation). The partition sum reads

$$Z = \sum_{\{S_i\}} \exp(-\beta H) \,.$$

The price to pay is to introduce a (Hubbard-Stratonovich) real variable x_i at each lattice site.

(c) It is now possible to exactly perform the summation over Ising variables $\sum_{\{S_i\}}$, which leads to a spatially local (potential) term in x_j. Then perform the variable transformation $\phi_i = \frac{1}{\sqrt{2}} \sum_j K_{ij}^{-1} x_j$ where $K_{ij} = \beta J_{ij}$ and x_j are the fields introduced in part (b).

(d) Now expand the exponent $H_{\text{eff}}[\phi_j]$ in $Z \sim \int \mathcal{D}[\phi_j] \exp(-\beta H_{\text{eff}}[\phi_j])$ up to quartic order in ϕ_j and then go to momentum space. Here, $\mathcal{D}[\phi_j]$ denotes the integration measure. Explicitly calculate the coupling function $J(\boldsymbol{k})$ in momentum space (assuming translational invariance $J_{ij} = J_{\boldsymbol{r}=\boldsymbol{r}_i-\boldsymbol{r}_j}$)

$$J(\boldsymbol{k}) = \sum_{\boldsymbol{r}} J_{\boldsymbol{r}} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}$$

assuming ferromagnetic nearest-neighbor coupling on a d-dimensional cubic lattice.

(e) Expand $J(\boldsymbol{k})$ up to second order in $\boldsymbol{k}^2$, and identify the temperature T_c at which the (T-dependent) constant in front of the $\boldsymbol{k}$-independent part of the quadratic term vanishes and changes sign.

Then expand the coefficient in front of the quadratic $\boldsymbol{k}^2$ term as well as the quartic term to lowest order in $T - T_c$ to bring the expression into the form

$$H_{\text{eff}}[\phi_{\boldsymbol{k}}] = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\phi_{\boldsymbol{k}}\left[a_0(T-T_c)+b\boldsymbol{k}^2\right]\phi_{-\boldsymbol{k}}+\frac{u}{4}\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\phi_{\boldsymbol{k}_1}\phi_{\boldsymbol{k}_2}\phi_{\boldsymbol{k}_3}\phi_{-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3}\,.$$

Identify the constants T_c, a_0, b and u.

(f) Transform back to real space to obtain the real-space form of $H_{\text{eff}}[\phi_{\boldsymbol{x}}]$. How is this expression related to Landau theory ?

2. Field theory for the three-state Potts model (10 + 10 + 10 = 30 Punkte)

An extension of the Ising model is the Potts model, which assumes that there are q states at each lattice site and the energy of two sites depends only on whether they are in the same state or not. The Hamiltonian reads

$$H = -\frac{J}{2}\sum_{\langle i,j\rangle}\left(q\delta_{q_i,q_j} - 1\right)$$

with $q_i \in \{1, \ldots, q\}$ and $\langle i, j\rangle$ denotes the sum over nearest-neighbors. We want to derive the mean-field theory for the $q = 3$ state Potts model using the same prescription as in exercise 1.

(a) Show that one can represent the energy of this system by $E = \sum_{i,j} J_{ij}\boldsymbol{s}_i \cdot \boldsymbol{s}_j$ with $\boldsymbol{s}_i$ being a two-dimensional normalized vector at site i. Give an explicit representation of these vectors $\boldsymbol{s}_i$.

(b) Following the same prescription as in exercise 1 derive the effective field theory up to quartic order in the Hubbard-Stratonovich fields starting from the microscopic Hamiltonian $H = \sum_{i,j} J_{ij}\boldsymbol{s}_i \cdot \boldsymbol{s}_j$ with $\boldsymbol{s}_i$ being one of the three possible vectors at site i.

(c) Check to each order in the fields that the terms in $H_{\text{eff}}[\phi_{\boldsymbol{x}}]$ obey the symmetries of the system.