Feldtheorie der Kondensierten Materie, RG und Quantenkritikalität SS 2013

Prof. Dr. J. Schmalian
Dr. P. Orth

Blatt 02
Besprechung 07.05.2013

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1. Coherent state path integral for bosons (15 + 15 = 30 Punkte)

Derive the coherent state path integral for the partition function

$$Z = \mathrm{Tr} e^{-\beta(H-\mu N)}$$

of the bosonic Hamiltonian

$$H = \sum_\alpha \epsilon_\alpha b_\alpha^\dagger b_\alpha + V(b_\alpha^\dagger, b_\alpha)\,.$$

Here, b_α is a bosonic creation operator with index α labeling a single-particle basis of the quadratic part of the Hamiltonian, V is a normal ordered boson-boson interaction term, and $N = \sum_\alpha b_\alpha^\dagger b_\alpha$. Decompose the imaginary time path into small time slices of duration $\delta\tau = \hbar\beta/N$ with $N \gg 1$, write $e^{-\beta H} = [e^{-(1/\hbar)\delta\tau H}]^N$ and use the Suzuki-Trotter decomposition $e^{A+B} = \lim_{N\to\infty}(e^{A/N}e^{B/N})^N$. Between different time slices insert resolutions if identity in terms of the bosonic coherent states

$$1 = \prod_\alpha \int \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} |\phi\rangle\langle\phi|\,,$$

where $\int \frac{d\phi_\alpha^* d\phi_\alpha}{2\pi i} = \int_{-\infty}^{\infty} \frac{d(\mathrm{Re}\phi_\alpha)d(\mathrm{Im}\phi_\alpha)}{\pi}$. Note that $b_\alpha|\phi\rangle = \phi_\alpha|\phi\rangle$ and $\langle\phi|b_\alpha^\dagger = \langle\phi|\phi_\alpha^*$.

(a) Derive the explicit form of the partition function Z in the discrete time representation. Then perform the continuum limit.

(b) The time-ordered Green's function is defined as $G(\tau_1,\tau_2) = \left\langle T\, b_\alpha(\tau_1) b_\alpha^\dagger(\tau_2)\right\rangle$ where $b_\alpha(\tau) = e^{\tau(H-\mu N)} b_\alpha e^{-\tau(H-\mu N)}$ and $b_\alpha^\dagger(\tau) = e^{\tau(H-\mu N)} b_\alpha^\dagger e^{-\tau(H-\mu N)}$ are in the imaginary-time Heisenberg representation. Show that using coherent state functional integrals it becomes the automatically time-ordered expression

$$G(\tau_1,\tau_2) = \frac{1}{Z}\int \mathcal{D}\big[\phi_\alpha^\dagger, \phi_\alpha\big]\phi(\tau_1)\phi^*(\tau_2)e^{-S}\,,$$

where $S = \int_0^\beta d\tau \sum_\alpha \phi_\alpha^*(\tau)\big(\frac{\partial}{\partial\tau} - \mu\big)\phi_\alpha(\tau) + H[\phi_\alpha^*(\tau), \phi_\alpha(\tau)]$. Since path integral expressions are automatically time-ordered, there is no need to explicitly include the time-ordering operator.

2. Quantum-to-classical mapping of quantum rotor model (15 + 15 = 30 Punkte)

Consider the Hamiltonian of a one-dimensional chain of superconducting metallic grains which are connected by Josephson junctions

$$H = \frac{C}{2}\sum_j \hat{V}_j^2 - E_j \cos(\hat\theta_j - \hat\theta_{j+1})\,.$$

Here, $\hat{\theta}_j$ represents the phase on the j-grain and $V_j = -i(2e/C)(\partial/\partial\hat{\theta}_j)$ is the operator conjugate to the phase $\hat{\theta}_j$ and represents the voltage on the jth Josephson junction.

(a) Derive the corresponding classical model by employing the Suzuki-Trotter decomposition, and inserting a complete basis of states at each time step.

(b) Use the Poisson formula

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi \; g(\phi) e^{-2\pi i \phi m}$$

to bring the factor stemming from the kinetic energy to the form

$$A \sum_{m=-\infty}^{\infty} e^{-\hbar/(2E_C \delta\tau)(\theta + 2\pi m)^2} \, ,$$

with constant A. This periodic sequence of very narrow Gaussians is (up to an irrelevant prefactor) the Villain approximation to $e^{\hbar/(E_C \delta\tau)\cos\theta}$. Show that one obtains an isotropic classical model by choosing the short-time cutoff as the inverse Josephson plasma frequency $\delta\tau = \hbar/\sqrt{E_C E_J}$.

3. Renormalization group of ϕ^3-theory and critical exponents (10 + 30 = 40 Punkte)

Consider the classical ϕ^3-theory determined by the effective Hamiltonian

$$H = \frac{1}{2}\int d^d x \Big(b(\nabla\phi)^2 + r_0 \phi^2 + c\phi^3 \Big)$$

with $r_0 = a(T - T_0)$ and $b, c > 0$. Do not worry about the fact that the Hamiltonian is not bounded from below, since one can add higher order terms in ϕ to remedy this fact. In the following we study the theory close to the phase transition, where these additional terms are irrelevant in the renormalization group sense.

(a) Determine the upper critical dimension d_u of the theory.

(b) Derive the renormalization group equations using the Wilson procedure of successively integrating out thin shells of fields with large momenta. Calculate the critical exponents close to the upper critical dimension in an expansion in the small quantity $\epsilon = d_u - d$, where d denotes the dimensionality of the theory. What is the value of the the anomalous exponent η, and how does it compare with the result of a ϕ^4-theory ?