Feldtheorie der Kondensierten Materie, RG und Quantenkritikalität SS 2013

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1. Renormalization group of ϕ^3-theory and critical exponents (20 + 30 = 50 Punkte)

Consider the classical ϕ^3-theory determined by the effective Hamiltonian

$$H = \frac{1}{2}\int d^dx\Big(c(\nabla\phi)^2 + r_0\phi^2 + u\phi^3\Big)$$

with $r_0 = a(T - T_0)$ and $c, u > 0$. Do not worry about the fact that the Hamiltonian is not bounded from below, since one can add higher order terms in ϕ to remedy this fact. In the following we study the theory close to the phase transition, where these additional terms are irrelevant in the renormalization group sense.

(a) Determine the upper critical dimension d_u of the theory.

(b) Derive the renormalization group equations using the Wilson procedure of successively integrating out thin shells of fields with large momenta. Calculate the critical exponents close to the upper critical dimension in an expansion in the small quantity $\epsilon = d_u - d$, where d denotes the dimensionality of the theory. What is the value of the the anomalous exponent η, and how does it compare with the result of a ϕ^4-theory ?

2. Dangerously irrelevant variable (20 + 10 + 20 = 50 Punkte)

Consider the classical ϕ^4-theory determined by the effective Hamiltonian

$$H = \int d^dx\Big(\frac{c}{2}(\nabla\phi)^2 + \frac{r_0}{2}\phi^2 + \frac{u}{4}\phi^4 - h\phi\Big),$$

with $r_0 = a(T - T_c)/T_c = at$ and $c, u > 0$.

(a) Derive the critical exponents within Landau theory, which are correct above the upper critical dimension $d_u = 4$. For the (thermodynamic) exponents $\beta, \delta, \gamma, \alpha$ you can neglect the spatial fluctuations. For the (correlation) exponents η, ξ include the spatial fluctuations, and solve the Euler-Lagrange equation that determines the minimum of H in Fourier space.

(b) Now perform a lowest order (tree-level) RG analysis of H, *i.e*, simply take the term that only contains slow fields into account and perform momentum and field rescaling, to derive the scaling behavior of the (singular part) of the free energy density $f = F(r_0, h, u)/V$ with $F = \ln Z$ and $Z = \int \mathcal{D}\phi^{<}\mathcal{D}\phi^{>}e^{-\beta H}$:

$$f(r_0, h, u) = b^{-d} f[b^{y_t} r_0, b^{y_h} h, b^{y_u} u]\,,$$

where $f[r(l), h(l), u(l)] = F[r(l), h(l), u(l)]/V$ with $F[r(l), h(l), u(l)] = \int \mathcal{D}\phi^{<} e^{-\beta H_{\text{eff}}(\phi^{<})}$. Determine the exponents y_t, y_h, y_u by Confirm that $y_u < 0$ for $d > d_u$.

(c) Derive the scaling form of the magnetization

$$m(r_0, h, u) = b^{y_m} m[b^{y_t} r_0, b^{y_h} h, b^{y_u} u]\,,$$

i.e., determine y_m. Then set $h = 0$ and $b = |t|^{-1/y_t}$. Noting that the last argument of the scaling function becomes small for $y_u < 0$ you may be tempted to set $u = 0$ to obtain the exponent β. What result do you obtain this way, and does it agree with the correct mean-field exponent β ?

To see what goes wrong show explicitly that within Landau theory $m(-1, 0, \tilde{u}) \propto \tilde{u}^{-1/2}$. Use this and the scaling form of m to obtain the correct exponent β. Note that the existence of a dangerously irrelevant variable for $d > d_u$ like u here imples the violation of hyperscaling, *i.e*, the scaling relations that contain the dimensionality d explicitly are violated above $d > d_u$. The mean-field exponents obey hyperscaling only exactly at $d = d_u$.