Sommer-Semester 2011

Moderne Theoretische Physik III

Statistische Physik

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Institut für Theorie der Kondensierten Materie

Di 09:45-11:15, Lehmann HS 022, Geb 30.22
Do 09:45-11:15, Lehmann HS 022, Geb 30.22

http://www.tkm.kit.edu/lehre/ss2011_1017.php

Ideales Bose-Gas

Der Zustand ist charakterisiert lediglich durch n_λ

$$N = \sum_\lambda n_\lambda \qquad E = \sum_\lambda n_\lambda \epsilon_\lambda \qquad n_\lambda = 0, 1, 2, \ldots, \infty$$

Bosonen

$$Z_G(T, V, \mu) = \sum_n e^{-\beta(E_n - \mu N_n)} = \sum_{\{n_\lambda\}} e^{-\beta \sum_\lambda n_\lambda (\epsilon_\lambda - \mu)} \Rightarrow Z_G = \prod_\lambda Z_\lambda$$

$$Z_\lambda \equiv \sum_{n_\lambda = 0}^{\infty} e^{-\beta(\epsilon_\lambda - \mu) n_\lambda} = \frac{1}{1 - e^{-\beta(\epsilon_\lambda - \mu)}}$$

$\forall \lambda$ gilt $\epsilon_\lambda \geq \mu$

Ideales Bose-Gas

Allgemeine Relationen

$$Z_G(T,V,\mu) = \prod_\lambda \left[\frac{1}{1-e^{-\beta(\epsilon_\lambda-\mu)}}\right]$$

$$\Omega(T,V,\mu) = -k_\mathrm{B}T\ln Z_G(T,V,\mu) = k_\mathrm{B}T\sum_\lambda \ln\left[1-e^{-\beta(\epsilon_\lambda-\mu)}\right]$$

$$\langle N\rangle = -\frac{\partial\Omega}{\partial\mu}\Big|_{T,V} = \sum_\lambda \langle n_\lambda\rangle \qquad \langle n_\lambda\rangle = n_B(\epsilon_\lambda) = \frac{1}{e^{\beta(\epsilon_\lambda-\mu)}-1}$$

$$S = -\frac{\partial\Omega}{\partial T}\Big|_{V,\mu} = -k_\mathrm{B}\sum_\lambda \left[\langle n_\lambda\rangle\ln\langle n_\lambda\rangle - (1+\langle n_\lambda\rangle)\ln(1+\langle n_\lambda\rangle)\right]$$

$$U = \Omega + TS + \mu N = \sum_\lambda \epsilon_\lambda n_B(\epsilon_\lambda)$$

Ideales Bose-Gas

Freie nichtrelativistische Teilchen im Kasten

$$\Omega(T,V,\mu) = -k_{\mathrm{B}}T \ln Z_G(T,V,\mu) = k_{\mathrm{B}}T \sum_{\lambda} \ln \left[1 - e^{-\beta(\epsilon_\lambda - \mu)}\right]$$

$\lambda = \mathbf{p}, \sigma$ z.B. ^{7}Li, ^{23}Na,... Gesamtspin von Elektronen und Kern $\mathbf{F} = \mathbf{S} + \mathbf{I}$

z.B. $F = 1$

$$\epsilon_\lambda = \epsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} \qquad \nu(\epsilon) = \frac{m^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}$$

Zustandsdichte

Ideales Bose-Gas

$$\Omega(T,V,\mu) = -k_\mathrm{B}T\ln Z_G(T,V,\mu) = k_\mathrm{B}T\sum_\lambda \ln\left[1-e^{-\beta(\epsilon_\lambda-\mu)}\right]$$

$$\epsilon_\lambda = \epsilon_\mathbf{p} = \frac{\mathbf{p^2}}{2m} \qquad \nu(\epsilon) = \frac{m^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}$$ Zustandsdichte

$$z \equiv e^{\beta\mu}$$ Fugazität

$$\Omega(T,V,\mu) = k_\mathrm{B}T\,(2s+1)\,\ln[1-z] + k_\mathrm{B}T(2s+1)V\int_0^\infty d\epsilon\,\nu(\epsilon)\ln\left[1-ze^{-\beta\epsilon}\right]$$

Der Beitrag von $\epsilon_\mathbf{p}=0$ muss separat behandelt werden um den Limes $\mu\to 0$ richtig zu behandeln!

Ideales Bose-Gas

$$z \equiv e^{\beta\mu}$$ **Fugazität**

$$\nu(\epsilon) = \frac{m^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}$$ **Zustandsdichte**

$$\Omega(T,V,\mu) = k_{\mathrm{B}}T\,(2s+1)\,\ln[1-z] + k_{\mathrm{B}}T(2s+1)V\int_0^\infty d\epsilon\,\nu(\epsilon)\ln\left[1-ze^{-\beta\epsilon}\right]$$

$$\Omega(T,V,\mu) = k_{\mathrm{B}}T\,(2s+1)\,\ln[1-z] - k_{\mathrm{B}}T(2s+1)\frac{V}{\lambda_T^3}\,g_{5/2}(z)$$

$$g_{5/2}(z) \equiv -\frac{4}{\sqrt{\pi}}\int_0^\infty dx\,x^2\ln(1-ze^{-x^2}) = \sum_{n=1}^\infty \frac{z^n}{n^{5/2}}$$

$$\lambda_{\mathrm{T}} = \sqrt{\frac{2\pi\hbar^2}{mk_{\mathrm{B}}T}}$$ **Thermische Wellenlänge**

Ideales Bose-Gas

$$z \equiv e^{\beta\mu}$$ **Fugazität** $$\nu(\epsilon) = \frac{m^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}$$ **Zustandsdichte**

$$\Omega(T,V,\mu) = k_\mathrm{B}T\,(2s+1)\,\ln[1-z] + k_\mathrm{B}T(2s+1)V\int_0^\infty d\epsilon\,\nu(\epsilon)\ln\left[1-ze^{-\beta\epsilon}\right]$$

$$N = -\frac{\partial\Omega}{\partial\mu}\Big|_{T,V} = (2s+1)\,\frac{z}{1-z} + (2s+1)V\int_0^\infty d\epsilon\,\nu(\epsilon)n_B(\epsilon)$$

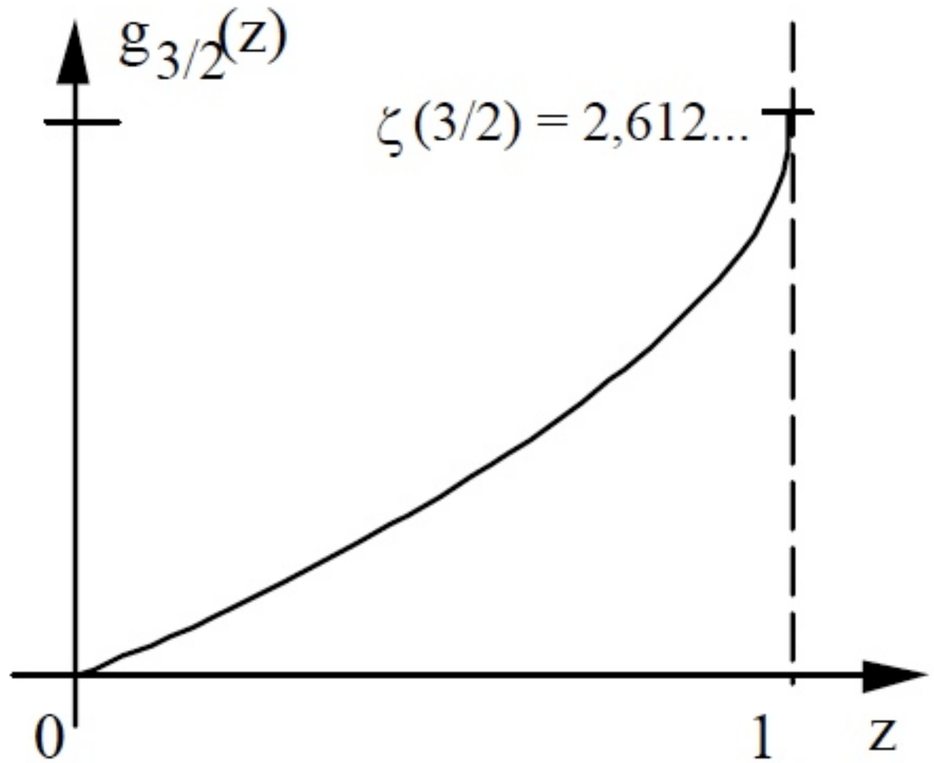
$$N = -\frac{\partial\Omega}{\partial\mu}\Big|_{T,V} = (2s+1)\,\frac{z}{1-z} + (2s+1)\frac{V}{\lambda_T^3}g_{3/2}(z)$$

$$g_{5/2}(z) = \sum_{n=1}^\infty \frac{z^n}{n^{5/2}} \qquad g_{3/2}(z) \equiv z\frac{\partial}{\partial z}g_{5/2}(z) = \sum_{n=1}^\infty \frac{z^n}{n^{3/2}} \qquad \lambda_\mathrm{T} = \sqrt{\frac{2\pi\hbar^2}{mk_\mathrm{B}T}}$$

Ideales Bose-Gas

$z \equiv e^{\beta\mu}$ **Fugazität**

$$N = -\frac{\partial\Omega}{\partial\mu}\Big|_{T,V} = (2s+1)\,\frac{z}{1-z} + (2s+1)\frac{V}{\lambda_T^3}g_{3/2}(z)$$



$$g_{5/2}(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^{5/2}} \qquad g_{3/2}(z) \equiv z\frac{\partial}{\partial z}g_{5/2}(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^{3/2}} \qquad \lambda_{\mathrm{T}} = \sqrt{\frac{2\pi\hbar^2}{mk_{\mathrm{B}}T}}$$

Ideales Bose-Gas

$z \equiv e^{\beta\mu}$ **Fugazität** $\nu(\epsilon) = \frac{m^{3/2}\epsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}$ **Zustandsdichte**

$$\Omega(T,V,\mu) = k_{\mathrm{B}}T\,(2s+1)\,\ln[1-z] - k_{\mathrm{B}}T(2s+1)\frac{V}{\lambda_T^3}\,g_{5/2}(z)$$

$$U = \Omega + TS + \mu N = \sum_\lambda \epsilon_\lambda n_B(\epsilon_\lambda)$$

Kein Beitrag von $\epsilon_{\mathbf{p}} = 0$

$$U(T,V,\mu) = \frac{3}{2}\,k_{\mathrm{B}}T(2s+1)\frac{V}{\lambda_T^3}\,g_{5/2}(z)$$

$g_{5/2}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{5/2}}$ $\lambda_{\mathrm{T}} = \sqrt{\frac{2\pi\hbar^2}{mk_{\mathrm{B}}T}}$

Ideales Bose-Gas

Hohe Temperaturen $n\lambda_T^3 \ll 1$

$$\Omega(T,V,\mu) = k_\mathrm{B}T\,(2s+1)\,\ln[1-z] - k_\mathrm{B}T(2s+1)\frac{V}{\lambda_T^3}\,g_{5/2}(z) \approx -k_\mathrm{B}T(2s+1)\frac{V}{\lambda_T^3}\,z$$

$$N = (2s+1)\,\frac{z}{1-z} + (2s+1)\frac{V}{\lambda_T^3}g_{3/2}(z) \approx (2s+1)\frac{V}{\lambda_T^3}\,z$$

$$U(T,V,\mu) = \frac{3}{2}\,k_\mathrm{B}T(2s+1)\frac{V}{\lambda_T^3}\,g_{5/2}(z) \approx \frac{3}{2}\,k_\mathrm{B}T(2s+1)\frac{V}{\lambda_T^3}\,z$$

$$PV = -\Omega = k_\mathrm{B}TN \qquad\qquad U = \frac{3}{2}\,k_\mathrm{B}TN$$

Ideales klassisches Gas

$$g_{5/2}(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^{5/2}} \qquad g_{3/2}(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^{3/2}} \qquad \lambda_\mathrm{T} = \sqrt{\frac{2\pi\hbar^2}{mk_\mathrm{B}T}}$$

Ideales Bose-Gas

Hohe Temperaturen $\quad n\lambda_T^3 \ll 1$

Korrekturen zum klassischen idealen Gas

$$N \approx (2s+1)\frac{V}{\lambda_T^3}\left(z + \frac{z^2}{2^{3/2}}\right) \qquad z \approx \frac{n\lambda_T^3}{(2s+1)}\left(1 - \frac{1}{2^{3/2}}\frac{n\lambda_T^3}{(2s+1)}\right)$$

$$\Omega \approx -k_\mathrm{B}T(2s+1)\frac{V}{\lambda_T^3}\left(z + \frac{z^2}{2^{5/2}}\right) \qquad PV \approx k_\mathrm{B}TN\left(1 - \frac{1}{2^{5/2}}\frac{n\lambda_T^3}{(2s+1)}\right)$$

$$U \approx \frac{3}{2}k_\mathrm{B}T(2s+1)\frac{V}{\lambda_T^3}\left(z + \frac{z^2}{2^{5/2}}\right) \qquad U \approx \frac{3}{2}k_\mathrm{B}TN\left(1 - \frac{1}{2^{5/2}}\frac{n\lambda_T^3}{(2s+1)}\right)$$

$$g_{5/2}(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^{5/2}} \qquad g_{3/2}(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^{3/2}} \qquad \lambda_\mathrm{T} = \sqrt{\frac{2\pi\hbar^2}{mk_\mathrm{B}T}}$$

Ideales Bose-Gas

$$\lambda_{\mathrm{T}} = \sqrt{\frac{2\pi\hbar^2}{mk_{\mathrm{B}}T}}$$

Hohe Temperaturen $n\lambda_T^3 \ll 1$

$$z \approx \frac{n\lambda_T^3}{(2s+1)} \ll 1 \quad \Rightarrow \quad \mu < -k_{\mathrm{B}}T$$

$$PV \approx k_{\mathrm{B}}TN\left(1 - \frac{1}{2^{5/2}}\frac{n\lambda_T^3}{(2s+1)}\right) < k_{\mathrm{B}}TN$$

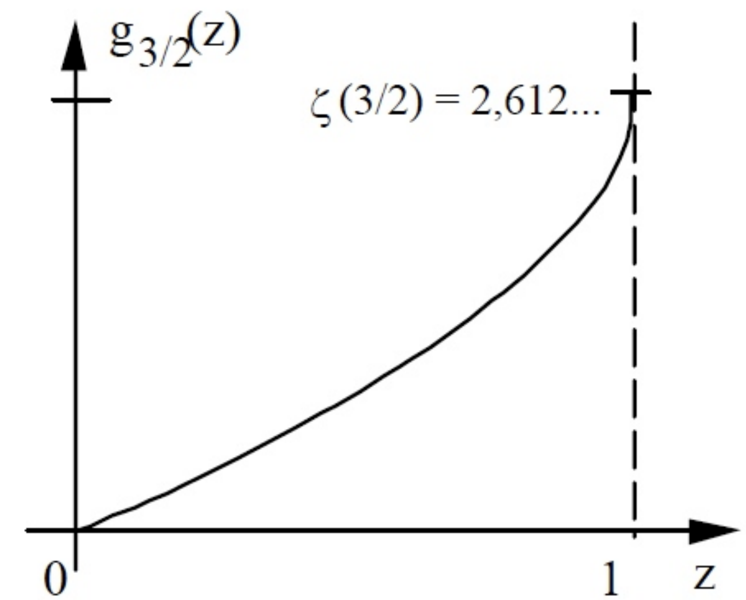
Bose-Statistik ~ Anziehung

Bose-Einstein-Kondensation

Tiefe Temperaturen $n\lambda_T^3 \gg 1$

$$N = (2s+1)\,\frac{z}{1-z} + (2s+1)\frac{V}{\lambda_T^3} g_{3/2}(z)$$

$$n \equiv \frac{N}{V} = \frac{N_0}{V} + (2s+1)\frac{1}{\lambda_T^3} g_{3/2}(z)$$



Teilchenzahl im Grundzustand

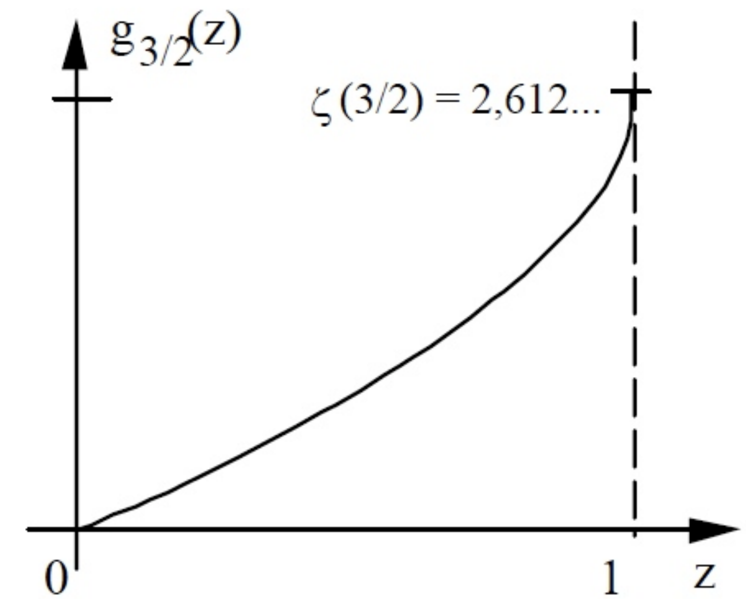
$$N_0 \equiv (2s+1)\,\frac{z}{1-z} = (2s+1) n_B(\epsilon = 0)$$

$$g_{3/2}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{3/2}} \qquad \lambda_{\mathrm{T}} = \sqrt{\frac{2\pi\hbar^2}{mk_{\mathrm{B}}T}}$$

Bose-Einstein-Kondensation

$$n \equiv \frac{N}{V} = \frac{N_0}{V} + (2s+1)\frac{1}{\lambda_T^3} g_{3/2}(z)$$

$$N_0 \equiv (2s+1)\,\frac{z}{1-z}$$



Für $$n < n_c \equiv (2s+1)\frac{1}{\lambda_T^3} g_{3/2}(1)$$

$$\lambda_\mathrm{T} = \sqrt{\frac{2\pi\hbar^2}{mk_\mathrm{B}T}}$$

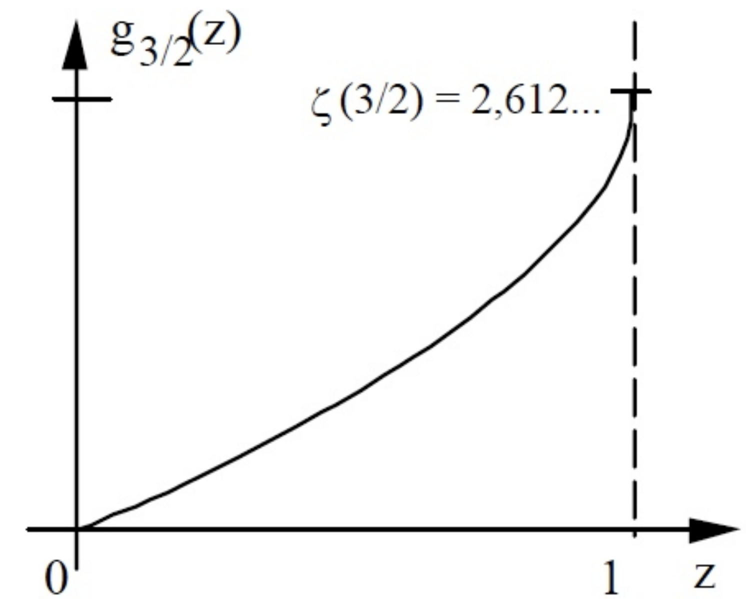
oder $$T > T_c \equiv \frac{2\pi\hbar^2}{mk_\mathrm{B}} \left(\frac{n}{(2s+1)g_{3/2}(1)}\right)^{2/3}$$

⇩ normales Gas

$$\lim_{V\to\infty} \frac{N_0}{V} = 0 \qquad \lim_{V\to\infty} z < 1 \qquad \lim_{V\to\infty} \mu < 0$$

Bose-Einstein-Kondensation

$$n \equiv \frac{N}{V} = \frac{N_0}{V} + (2s+1)\frac{1}{\lambda_T^3} g_{3/2}(z)$$



$$N_0 \equiv (2s+1)\,\frac{z}{1-z}$$

Für $\quad n > n_c \equiv (2s+1)\frac{1}{\lambda_T^3} g_{3/2}(1)$

oder $\quad T < T_c \equiv \frac{2\pi\hbar^2}{mk_\mathrm{B}} \left(\frac{n}{(2s+1)g_{3/2}(1)}\right)^{2/3}$

$$\lambda_\mathrm{T} = \sqrt{\frac{2\pi\hbar^2}{mk_\mathrm{B}T}}$$

Teil der Teilchen im Kondensat

$$\lim_{V\to\infty} \frac{N_0}{V} \equiv n_0 = n - n_c \qquad \lim_{V\to\infty} z = 1 \qquad \lim_{V\to\infty} \mu = 0$$

Bose-Einstein-Kondensation

Aus Wikipedia

This state of matter was first predicted by Satyendra Nath Bose and Albert Einstein in 1924–25. Bose first sent a paper to Einstein on the quantum statistics of light quanta (now called photons). Einstein was impressed, translated the paper himself from English to German and submitted it for Bose to the Zeitschrift für Physik which published it. Einstein then extended Bose's ideas to material particles (or matter) in two other papers.

The Einstein manuscript, believed to be lost, was found in a library at Leiden University in 2005.

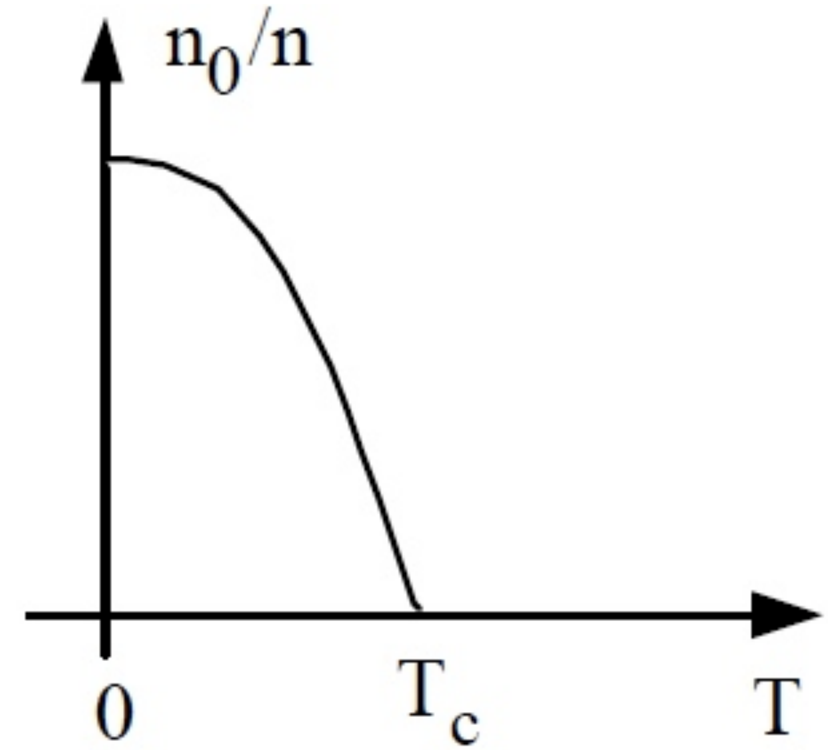
Bose-Einstein-Kondensation

Für $T < T_c \equiv \frac{2\pi\hbar^2}{mk_\mathrm{B}} \left(\frac{n}{(2s+1)g_{3/2}(1)} \right)^{2/3}$

$$n = n_0 + (2s+1)\frac{1}{\lambda_T^3} g_{3/2}(1)$$

$$\lambda_\mathrm{T} = \sqrt{\frac{2\pi\hbar^2}{mk_\mathrm{B}T}}$$

$$\frac{n_0}{n} = 1 - (2s+1)\frac{1}{n\lambda_T^3} g_{3/2}(1) = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$



The Nobel Prize in Physics 2001

"for the achievement of Bose-Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates"

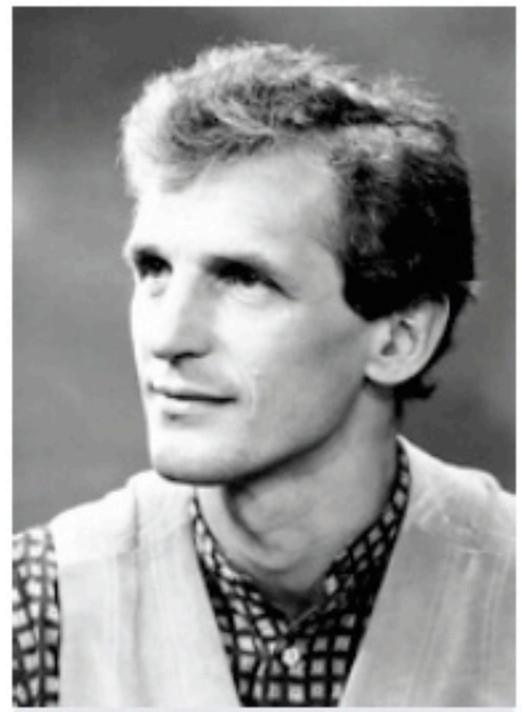
Eric A. Cornell

1/3 of the prize

USA

University of Colorado, JILA
Boulder, CO, USA

b. 1961



Wolfgang Ketterle

1/3 of the prize

Federal Republic of Germany

Massachusetts Institute of Technology (MIT)
Cambridge, MA, USA

b. 1957

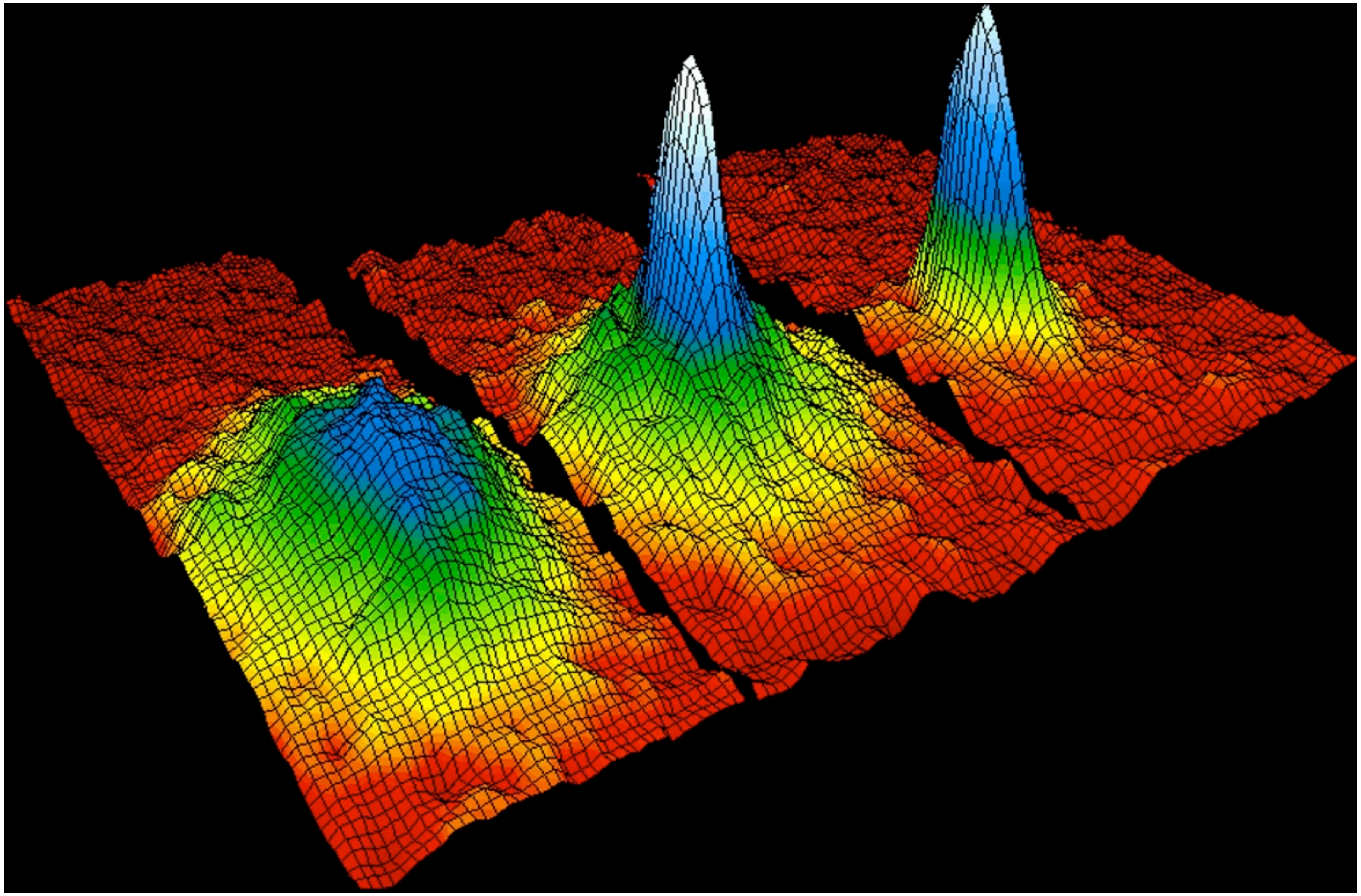
Carl E. Wieman

1/3 of the prize

USA

University of Colorado, JILA
Boulder, CO, USA

b. 1951



Geschwindigkeitsverteilung im BEC

Ideales Bose-Gas

Druck

$$\Omega(T,V,\mu) = k_\mathrm{B}T\,(2s+1)\,\ln[1-z] - k_\mathrm{B}T(2s+1)\frac{V}{\lambda_T^3}\,g_{5/2}(z)$$

$$P = -\frac{\Omega}{V} = -k_\mathrm{B}T\,(2s+1)\frac{\ln[1-z]}{V} + k_\mathrm{B}T(2s+1)\frac{1}{\lambda_T^3}\,g_{5/2}(z)$$

$$\lim_{V\to\infty}\frac{\ln[1-z]}{V} = 0 \quad \Rightarrow \quad P = k_\mathrm{B}T(2s+1)\frac{1}{\lambda_T^3}\,g_{5/2}(z)$$

$$g_{5/2}(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^{5/2}}$$

$$\lambda_\mathrm{T} = \sqrt{\frac{2\pi\hbar^2}{mk_\mathrm{B}T}}$$

Ideales Bose-Gas

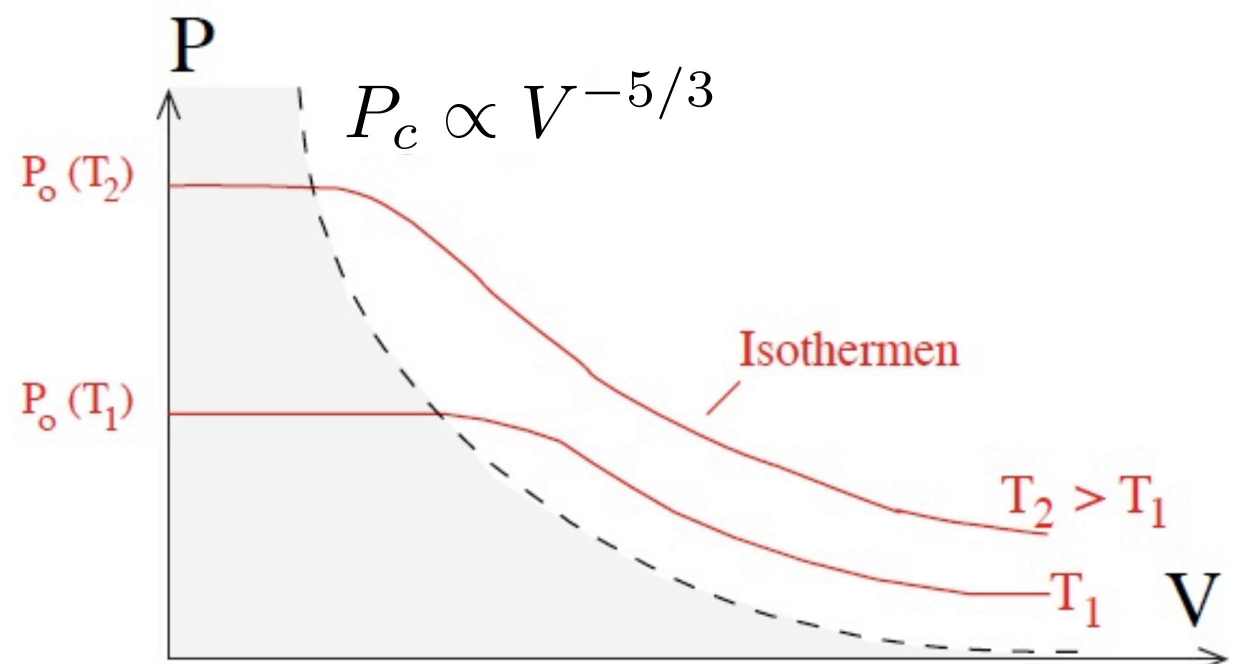
Druck

$$P = k_{\mathrm{B}} T (2s+1) \frac{1}{\lambda_T^3} \, g_{5/2}(z)$$

$$T_c \equiv \frac{2\pi\hbar^2}{m k_{\mathrm{B}}} \left(\frac{n}{(2s+1) g_{3/2}(1)} \right)^{2/3}$$

$$P_c = k_{\mathrm{B}} T_c (2s+1) \frac{1}{\lambda_{T_c}^3} \, g_{5/2}(1)$$

$$P_c \propto V^{-5/3}$$



$$P_c \propto n^{5/3} \propto V^{-5/3}$$

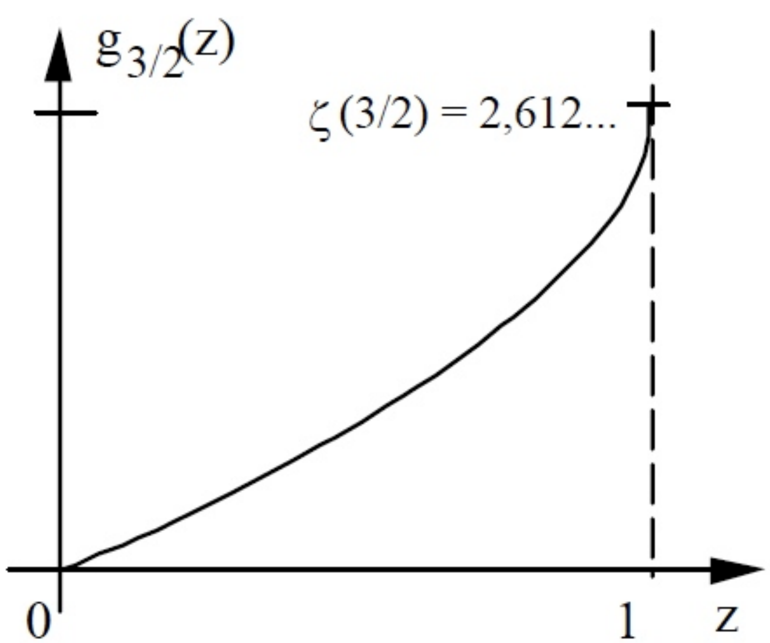
$$g_{5/2}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{5/2}}$$

$$\lambda_{\mathrm{T}} = \sqrt{\frac{2\pi\hbar^2}{m k_{\mathrm{B}} T}}$$

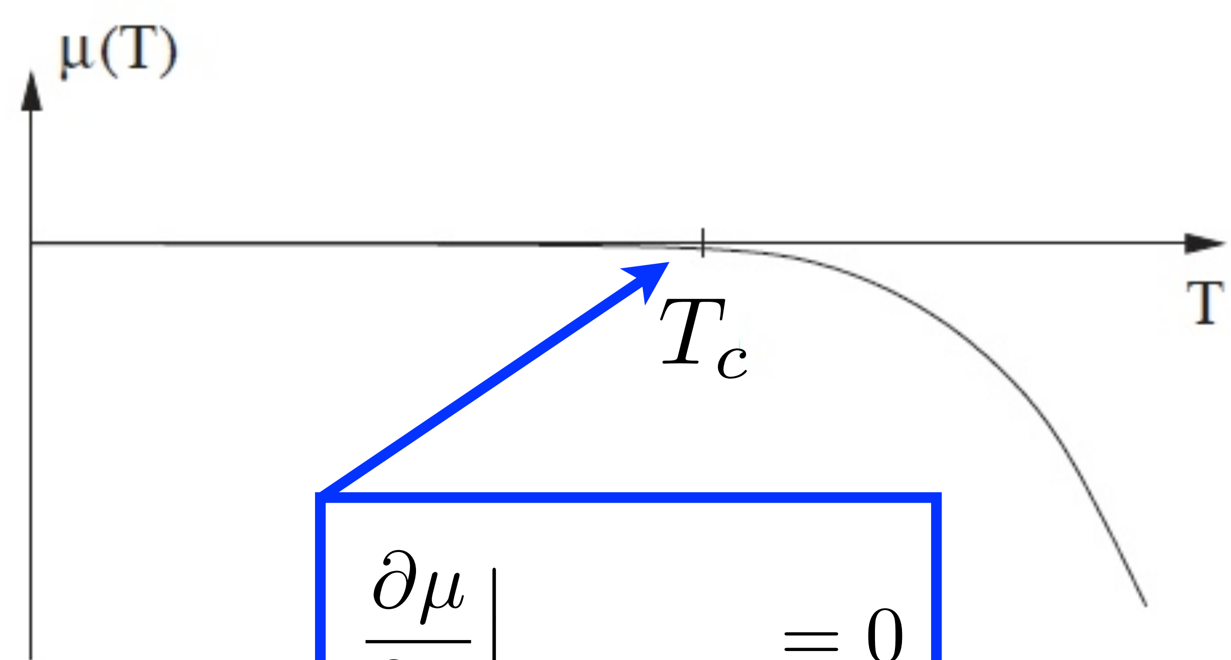
Bose-Einstein-Kondensation

Chemisches Potential

$$N = (2s+1)\,\frac{z}{1-z} + (2s+1)\,\frac{V}{\lambda_T^3}\,g_{3/2}(z)$$



$$g_{3/2}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{3/2}}$$



$$\left.\frac{\partial \mu}{\partial T}\right|_{T=T_c+0} = 0$$

$$\left.\frac{\partial^2 \mu}{\partial T^2}\right|_{T=T_c+0} < 0$$

Bose-Einstein-Kondensation

Entropie

$$\Omega(T,V,\mu) = k_\mathrm{B}T\,(2s+1)\,\ln[1-z] - k_\mathrm{B}T(2s+1)\frac{V}{\lambda_T^3}\,g_{5/2}(z)$$

$$N = (2s+1)\,\frac{z}{1-z} + (2s+1)\frac{V}{\lambda_T^3}g_{3/2}(z) \qquad \lim_{V\to\infty}\frac{\ln[1-z]}{V} = 0$$

für $T > T_c$ $\qquad S = -\frac{\partial\Omega}{\partial T}\Big|_{V,\mu} = \frac{5}{2}\,k_\mathrm{B}(2s+1)\frac{V}{\lambda_T^3}\,g_{5/2}(z) - k_\mathrm{B}N\ln z$

für $T < T_c$ $\qquad S = -\frac{\partial\Omega}{\partial T}\Big|_{V,\mu} = \frac{5}{2}\,k_\mathrm{B}(2s+1)\frac{V}{\lambda_T^3}\,g_{5/2}(1) \propto T^{3/2}$

3. Hauptsatz

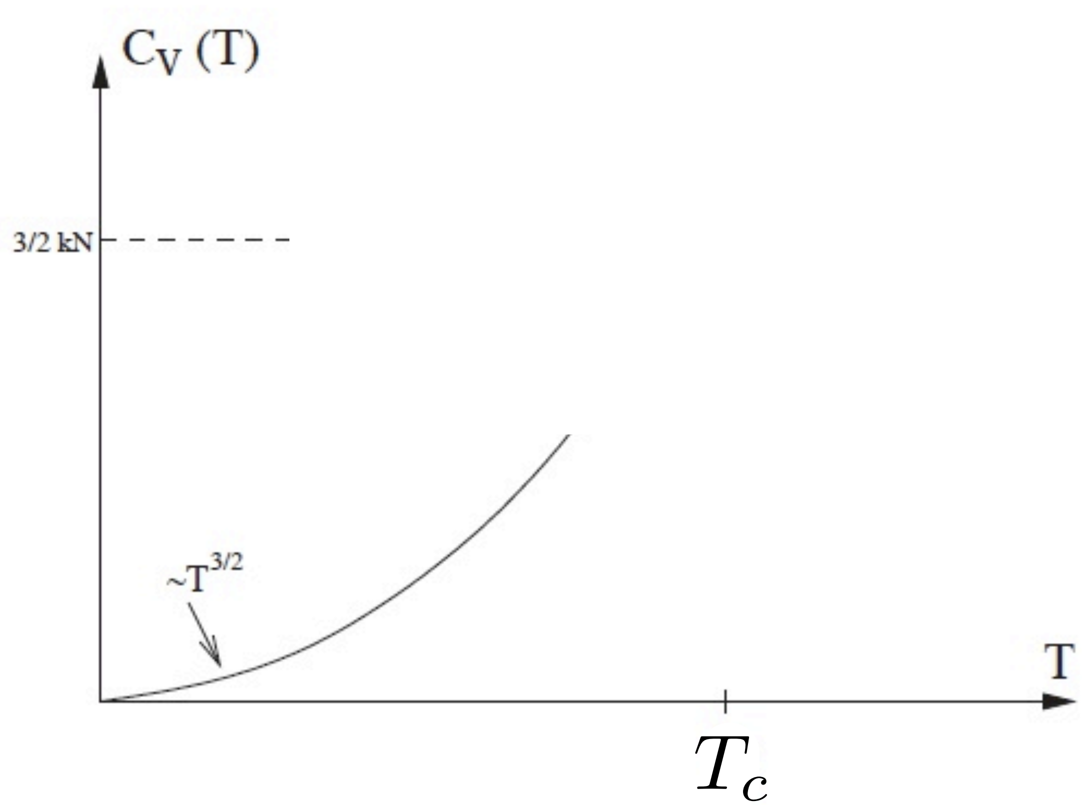
Bose-Einstein-Kondensation

Wärmekapazität

$$C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N} = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = T\left(\frac{\partial S(T,V,\mu(N,T))}{\partial T}\right)_{V,N}$$

für $T < T_c$ gilt $\mu = 0 = const.$ und $S = \frac{5}{2}\, k_\mathrm{B}(2s+1)\frac{V}{\lambda_T^3}\, g_{5/2}(1) \propto T^{3/2}$

für $T < T_c$ gilt $C_V = \frac{15}{4}\, k_\mathrm{B}(2s+1)\frac{V}{\lambda_T^3}\, g_{5/2}(1) \propto T^{3/2}$



Bose-Einstein-Kondensation

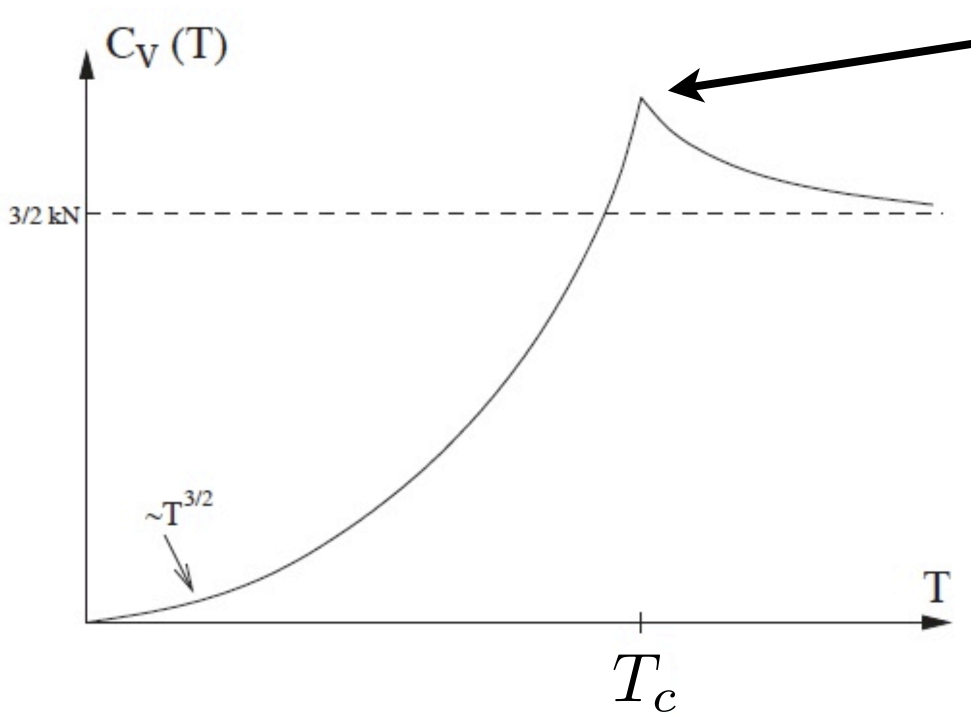
Wärmekapazität

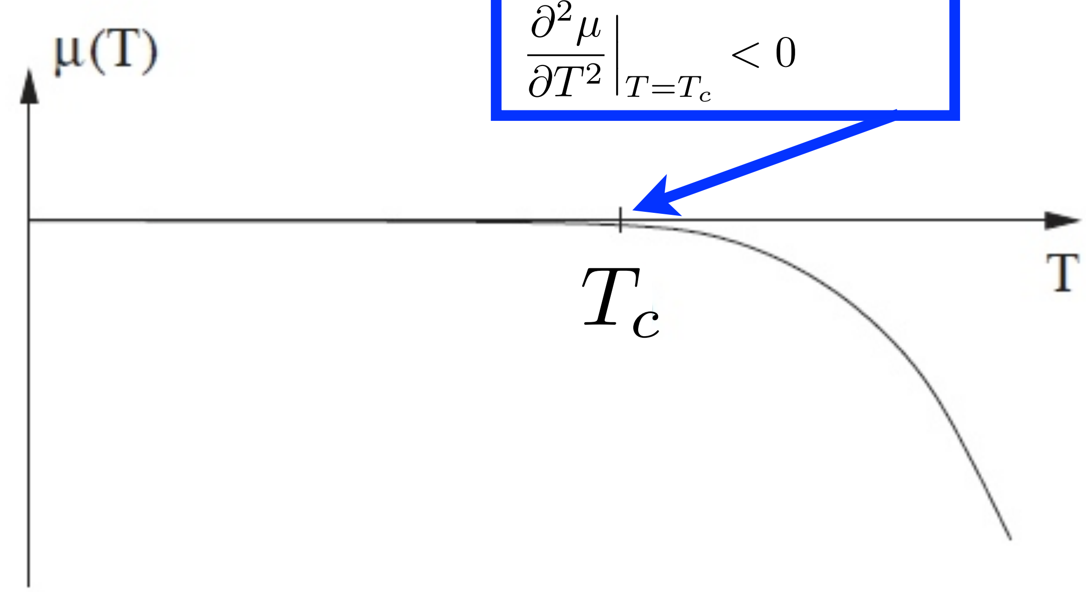
$$U(T,V,\mu) = \frac{3}{2}\, k_\mathrm{B} T (2s+1) \frac{V}{\lambda_T^3}\, g_{5/2}(z)$$

$$\begin{aligned} C_V &= \left(\frac{\partial U(T,V,\mu(N,T))}{\partial T}\right)_{V,N} \\ &= \frac{15}{4}\, k_\mathrm{B} T(2s+1)\frac{V}{\lambda_T^3}\, g_{5/2}(z) - \frac{3}{2}\left(\frac{\mu}{T}\right)\frac{V}{\lambda_T^3}\, g_{3/2}(z) + \frac{3}{2}\left(\frac{\partial \mu}{\partial T}\right)\frac{V}{\lambda_T^3}\, g_{3/2}(z) \end{aligned}$$

$$\left.\frac{\partial \mu}{\partial T}\right|_{T=T_c} = 0$$

$$\left.\frac{\partial^2 \mu}{\partial T^2}\right|_{T=T_c} < 0$$





Bose-Einstein-Kondensation

Erfolgt nicht in 2D oder 1D

$$N = \sum_\lambda \langle n_\lambda \rangle = \ldots + (2s+1)V \int_0^\infty d\epsilon \nu(\epsilon) n_B(\epsilon)$$

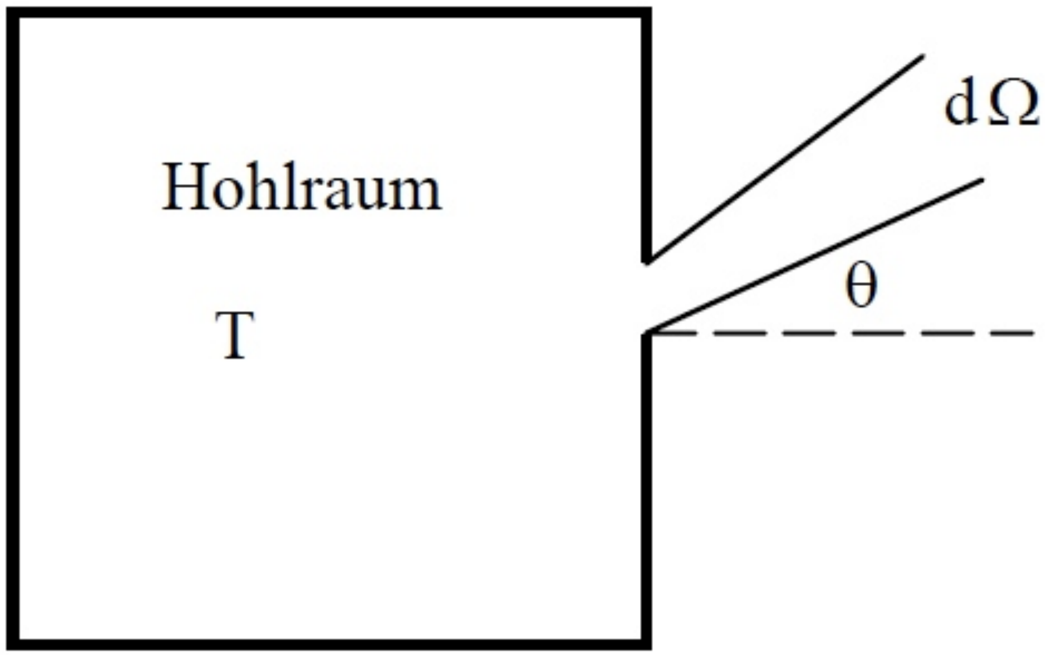
in 2D gilt $\nu(\epsilon) \propto \epsilon^0$

in 1D gilt $\nu(\epsilon) \propto \epsilon^{-1/2}$

Makroskopische Besetzung des Grundzustandes nicht nötig

Hohlraum-Strahlung

Photonen



nur transversale Photonen

$$\lambda = \mathbf{k}, \sigma \qquad \sigma = \pm 1$$

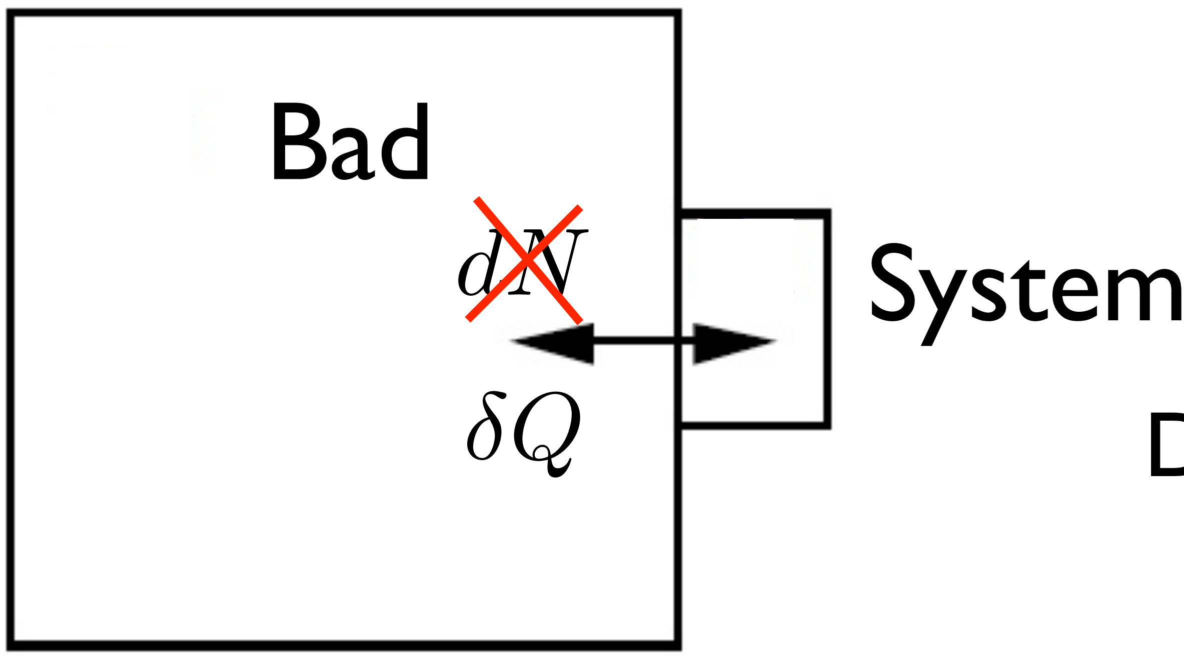
$$\mathbf{p} = \hbar \mathbf{k}$$

$$\epsilon_\lambda = \epsilon_{\mathbf{k}} = cp = c\hbar k$$

$$\sum_\lambda f(\epsilon) = 2V \int d\epsilon \nu(\epsilon) f(\epsilon) \qquad \nu(\epsilon) = \frac{4\pi\epsilon^2}{(2\pi\hbar c)^3}$$

Zahl der Photonen nicht erhalten $\Rightarrow$ $\mu = 0$

Großkanonische Gesamtheit alternative Herleitung



$$E = E^S + E^B \qquad N = N^S + N^B$$

$$E^S \ll E^B \qquad N^S \ll N^B$$

Das Gesamtsystem (System + Bad) ist mikrokanonisch

$$W_{n,m} = \frac{1}{\Delta N(E)} \text{ für } E < E_n^S + E_m^B < E + dE, \text{ sonst } 0$$

$$W_n = \sum_m W_{n,m} = \frac{\Delta N^B(E - E_n, N - N_n)}{\Delta N(E)}$$

$\Delta N(E)$ - Zahl der Zustände im Energiefenster $[E, E + dE]$

$$k_\mathrm{B} \ln W_n = k_\mathrm{B} \ln \left[\Delta N^B(E - E_n, N - N_n)\right] - k_\mathrm{B} \left[\Delta N(E)\right]$$

$$= S^B(E - E_n, N - N_n) - S = S^B(E, N) - E_n \frac{\partial S^B(E)}{\partial E} - N_n \frac{\partial S^B(E, N)}{\partial N} - S$$

$$= const. - \frac{E_n - \mu N_n}{T}$$

Photonen

Der Zustand ist charakterisiert lediglich durch n_λ

$$N = \sum_\lambda n_\lambda \qquad E = \sum_\lambda n_\lambda \epsilon_\lambda \qquad n_\lambda = 0, 1, 2, \ldots, \infty$$

Bosonen

kanonisch oder großkanonisch?

kanonisch: Wärme-Austausch durch Erzeugung und Vernichtung von Photonen

$$Z(T,V) = \sum_n e^{-\beta E_n} = \sum_{\{n_\lambda\}} e^{-\beta \sum_\lambda n_\lambda \epsilon_\lambda}$$

großkanonisch:

$$Z_G(T,V,\mu=0) = Z$$

$$Z = \prod_\lambda Z_\lambda$$

$$Z_\lambda \equiv \sum_{n_\lambda=0}^{\infty} e^{-\beta \epsilon_\lambda n_\lambda} = \frac{1}{1 - e^{-\beta \epsilon_\lambda}}$$

Photonen

$$F(T,V) = \Omega(T,V,\mu=0) = -k_{\mathrm{B}}T \ln Z_G(T,V,\mu=0) = k_{\mathrm{B}}T \sum_{\lambda} \ln\left[1 - e^{-\beta\epsilon_\lambda}\right]$$

$$\lambda = \mathbf{k}, \sigma \qquad \sigma = \pm 1 \qquad \epsilon_\lambda = \epsilon_{\mathbf{k}} = c\hbar k \qquad \nu(\epsilon) = \frac{4\pi\epsilon^2}{(2\pi\hbar c)^3}$$

Frequenz $\omega_{\mathbf{k}} \equiv \frac{\epsilon_{\mathbf{k}}}{\hbar} = ck \qquad \nu(\omega) = \frac{4\pi\omega^2}{(2\pi c)^3}$

$$\Omega = 2\, V k_{\mathrm{B}}T \int_0^\infty d\omega\, \nu(\omega) \ln\left[1 - e^{-\beta\hbar\omega}\right] \qquad \int_0^\infty dx\, x^2 \ln\left[1 - e^{-x}\right] = -\frac{\pi^4}{45}$$

$$F = \Omega = -V\,\frac{\pi^2}{45}\,\frac{(k_{\mathrm{B}}T)^4}{(c\hbar)^3}$$

Photonen

$$F = \Omega = -V \frac{\pi^2}{45} \frac{(k_\mathrm{B}T)^4}{(c\hbar)^3}$$

$$S = -\frac{\partial F}{\partial T}\Big|_V = -4\,\frac{F}{T} = k_\mathrm{B}V\,\frac{4\pi^2}{45}\,\frac{(k_\mathrm{B}T)^3}{(c\hbar)^3}$$

$$U = F + TS = -3F$$

$$C_V = T\frac{\partial S}{\partial T}\Big|_V = k_\mathrm{B}V\,\frac{4\pi^2}{15}\,\frac{(k_\mathrm{B}T)^3}{(c\hbar)^3}$$

sinnvolle Größe

$$P = -\frac{\Omega}{V} = \frac{\pi^2}{45}\,\frac{(k_\mathrm{B}T)^4}{(c\hbar)^3}$$

Strahlungsdruck

Plank'sche Strahlungsformel

$$\langle n_{\mathbf{k},\sigma} \rangle = \frac{1}{e^{\beta\hbar\omega_{\mathbf{k}}} - 1}$$

Zahl der Photonen im Frequenz-Fenster $\quad n_\omega d\omega = \langle n_{\mathbf{k},\sigma} \rangle 2V \, \frac{4\pi k^2 dk}{(2\pi)^3}$

Strahlungsenergie im Frequenz-Fenster pro Volumen

$$u(\omega, T) d\omega \equiv \frac{1}{V} n_\omega \hbar\omega d\omega$$

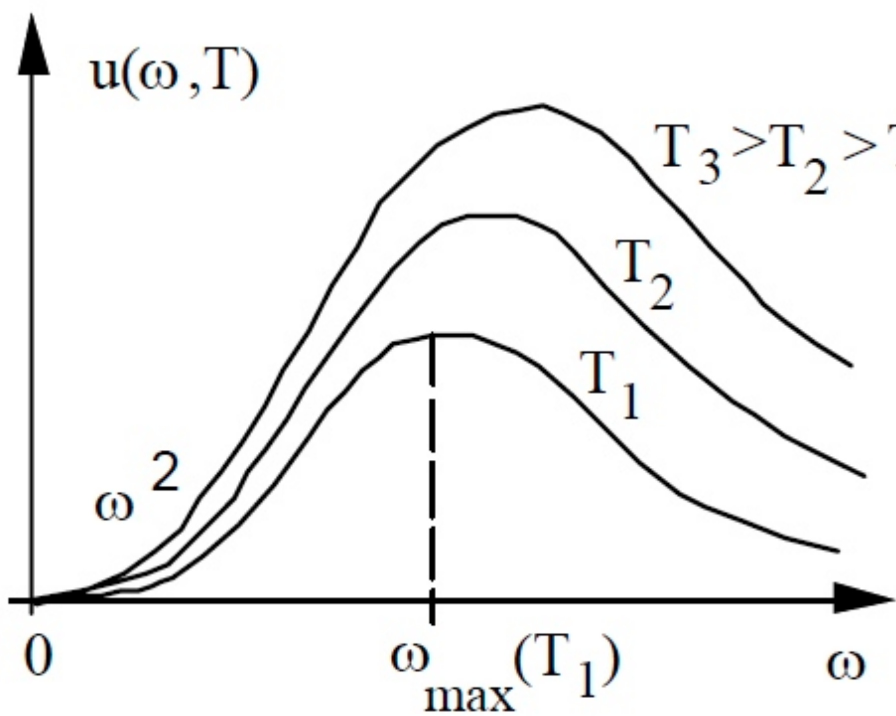
$$u(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$

Plank-Formel
(1900)

Plank'sche Strahlungsformel

Strahlungsenergie (pro Frequenz, pro Volumen)

$$u(\omega, T) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}$$ Plank-Formel



$$\hbar\omega_{\max} = 2,822 k_B T$$

Wien'sches Verschiebungsgesetz

für $\hbar\omega \ll k_B T$

$$u(\omega, T) = \frac{1}{\pi^2 c^3} k_B T \, \omega^2$$

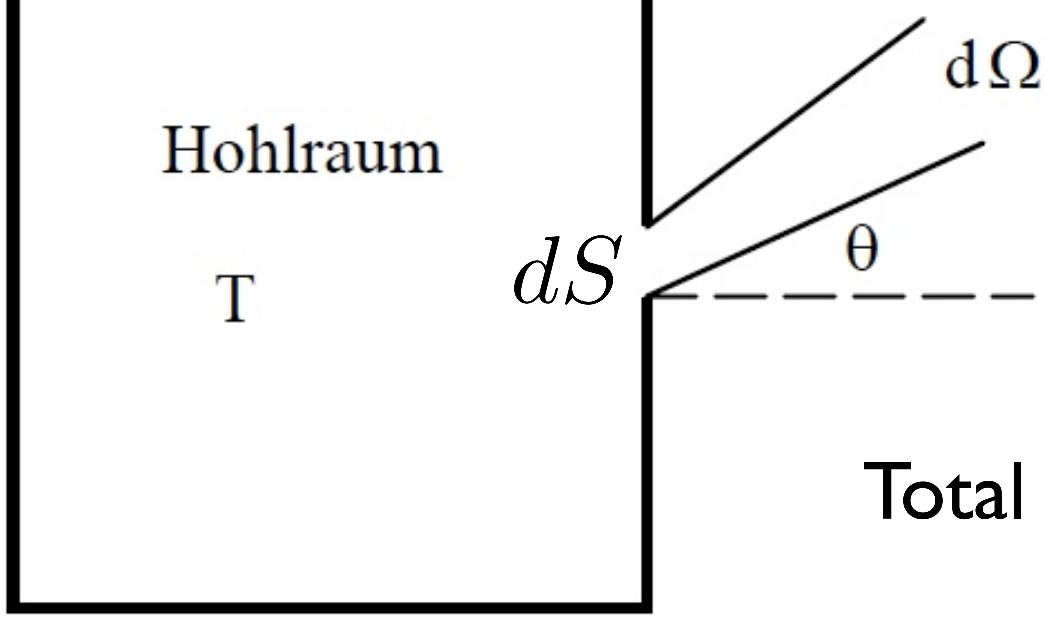
Reyleigh-Jeans-Gesetz

für $\hbar\omega \gg k_B T$

$$u(\omega, T) = \frac{1}{\pi^2 c^3} \hbar\omega^3 \, e^{-\beta\hbar\omega}$$

Wien'sches Gesetz

Hohlraum-Strahlung



Abgestrahlte Leistung

$$dI(\theta) = u(\omega, T)\, c\, d\omega\, d\Omega\, (dS \cos\theta)$$

Total abgestrahlte Leistung pro Fläche

$$\frac{dI}{dS} = c \int d\omega\, u(\omega, T) \int_{\text{Halbraum}} d\Omega \cos\theta$$

$$\frac{dI}{dS} = \frac{c}{4} \int d\omega\, u(\omega, T) = \sigma T^4$$

Stefan'sches Gesetz

$$\sigma = \frac{\pi^2 k_{\mathrm{B}}^4}{60 \hbar^3 c^2}$$

Akustische Phononen

Schall-Wellen

$\lambda = \mathbf{k}, \sigma$ longitudinale und 2x transversale Photonen $\sigma = l, t_1, t_2$

$\mathbf{p} = \hbar \mathbf{k}$ $\epsilon_\lambda = \epsilon_\mathbf{k} = cp = c\hbar k = \hbar\omega_\mathbf{k}$ c - Schall-Geschwindigkeit

Debye-Modell

$$\nu(\omega) = \frac{4\pi\omega^2}{(2\pi c)^3} \quad \text{für } \omega < \omega_D$$

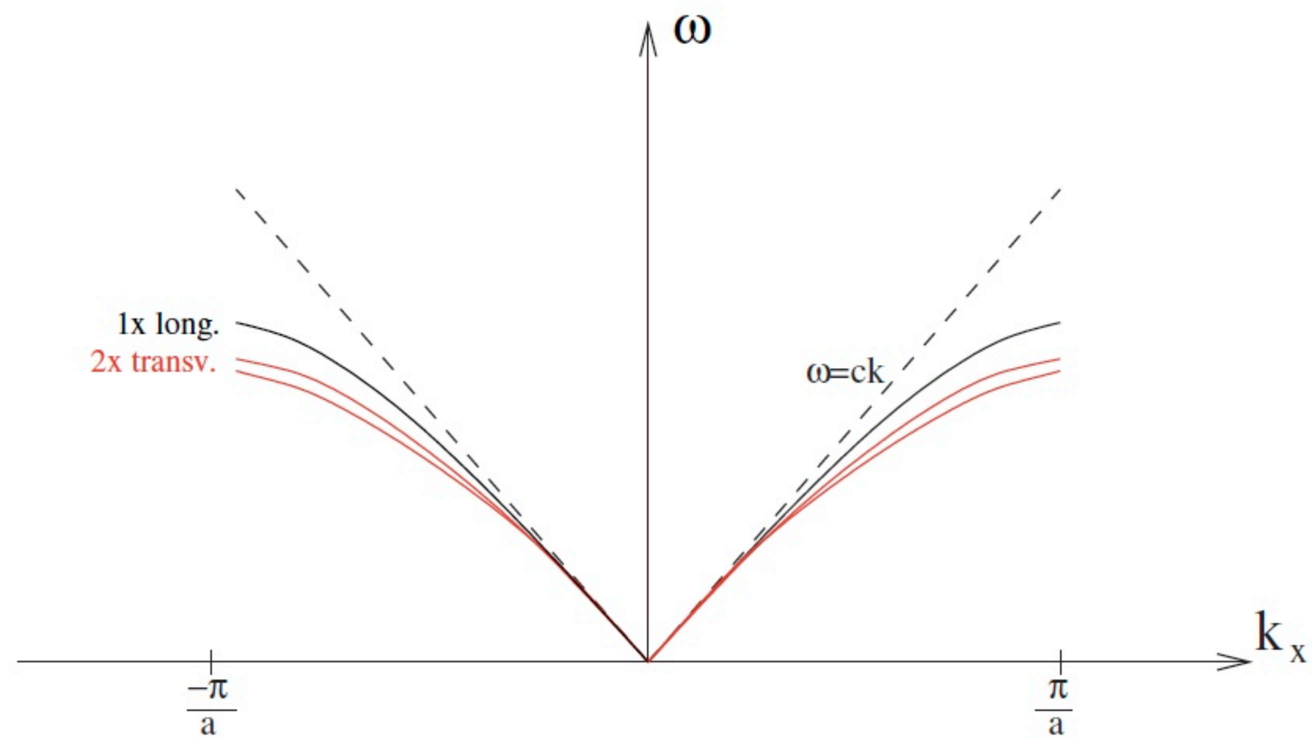
$$\nu(\omega) = 0 \quad \text{für } \omega > \omega_D$$

$$\omega_D = ck_D \sim \frac{c\pi}{a}$$

a - Gitterkonstante

Normierung, N Atome (Elementarzellen)

$$3\int_0^{\omega_D} d\omega\,\nu(\omega) = 3\,\frac{N}{V}$$



Akustische Phononen

Debye-Modell

$$\nu(\omega) = \frac{4\pi\omega^2}{(2\pi c)^3} \qquad \text{für } \omega < \omega_D$$

$$\nu(\omega) = 0 \qquad \text{für } \omega > \omega_D$$

$$\Omega = 3\, V k_\mathrm{B} T \int_0^{\omega_D} d\omega\, \nu(\omega) \ln\left[1 - e^{-\beta\hbar\omega}\right]$$

$$\theta_D \equiv \frac{\hbar\omega_D}{k_\mathrm{B}}$$

Debye-Temperatur

Übung

$$C_V \propto \frac{T^3}{\theta_D^3} \qquad \text{für } T \ll \theta_D$$

$$C_V = 3 N k_\mathrm{B} \qquad \text{für } T \gg \theta_D$$