Karlsruher Institut für Technologie — Institut für Theorie der Kondensierten Materie

Übungen zur Theorie der Kondensierten Materie II SS 16

Prof. J. Schmalian
B. Jeevanesan, M. S. Scheurer

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1. Kitaev chain and Majorana operators (9 Points)

Let us reconsider the Kitaev chain model,

$$H = -\sum_{j=1}^{N-1}\left[t\,c_j^\dagger c_{j+1} + \Delta\,c_j c_{j+1} + \text{H.c.}\right] - \mu\sum_{j=1}^{N}\left(c_j^\dagger c_j - \frac{1}{2}\right), \tag{1}$$

$t, \Delta, \mu \in \mathbb{R}$, that we encountered last week when performing a Jordan-Wigner transformation to the 1D XY model. We have already calculated the spectrum of this Hamiltonian in the case of periodic boundary conditions. We have thus analyzed a ring rather than a chain. In this exercise, we will investigate what happens for a finite chain with edges. For this purpose, it is convenient to introduce two Hermitian operators (also known as Majorana[1] operators) $\gamma_{jA} = \gamma_{jA}^\dagger$ and $\gamma_{jB} = \gamma_{jB}^\dagger$ per site j and write

$$c_j = \frac{1}{\sqrt{2}}\left(\gamma_{jA} + i\gamma_{jB}\right), \qquad c_j^\dagger = \frac{1}{\sqrt{2}}\left(\gamma_{jA} - i\gamma_{jB}\right). \tag{2}$$

(a) Determine the anticommutation relations satisfied by the Majorana operators. (2 Pts.)

(b) Represent the Kitaev model (1) in terms of Majorana operators. (2 Pts.)

(c) For the special case $\Delta = t$, $\mu = 0$, the Hamiltonian assumes a particularly simple form in terms of the new operators

$$b_j = \frac{1}{\sqrt{2}}\left(\gamma_{jA} + i\gamma_{j+1B}\right), \qquad b_j^\dagger = \frac{1}{\sqrt{2}}\left(\gamma_{jA} - i\gamma_{j+1B}\right). \tag{3}$$

Show that b_j and $b_j^\dagger$ satisfy the usual fermionic anticommutation relations. Rewrite H for $\Delta = t$, $\mu = 0$ in terms of b_j and $b_j^\dagger$ and determine the spectrum of the system. What is the corresponding magnetic model? Check that the spectrum of the latter agrees with the spectrum you have calculated from the Kitaev model. (4 Pts.)

(d) Taking a closer look at the Hamiltonian expressed in terms of Majorana operators in the limit $\Delta = t$, $\mu = 0$ reveals that there are exactly two out of the $2N$ Majorana operators introduced in Eq. (2) that do not enter the Hamiltonian. Where are these operators located spatially? Show that this leads to a two-fold degeneracy of the spectrum. (*Hint:* Use these two Majorana operators to construct a single ordinary fermionic operator in the same way as in Eq. (2)). Explain the degeneracy in the corresponding magnetic model. (1 Pt.)

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[1] Named after the Italian physicist E. Majorana (1906 – 1959).

2. Matsubara summation (9 Points)

In this exercise, we will learn how sums of the form

$$S = T \sum_{n\in\mathbb{Z}} h(i\omega_n) \tag{4}$$

with $\omega_n = 2n\pi T$ for bosons ($\eta = -1$ in the following) and $\omega_n = (2n+1)\pi T$ for fermions ($\eta = +1$) can be very efficiently evaluated at arbitrary temperature T. This is important as we will encounter expressions of the form (4) very frequently in the remainder of the lecture course.

(a) As a first step, determine the poles of the Fermi ($\eta = +1$) and Bose ($\eta = -1$) function,

$$n_\eta(z) = \frac{1}{e^{\beta z} + \eta}, \qquad \beta = T^{-1}, \tag{5}$$

and the associated residues. (2 Pts.)

(b) With this in mind, show that one can write

$$S = \frac{-\eta}{2\pi i} \oint_{\mathcal{C}} \mathrm{d}z\, n_\eta(z) h(z), \tag{6}$$

where the contour $\mathcal{C}$ encloses the infinite set of points $\{i\omega_n | n \in \mathbb{Z}\}$ in a counter-clockwise manner and $h(z)$ is analytic in the domain bound by $\mathcal{C}$. (1 Pt.)

(c) As a first example, let $h(z) = f(z)e^{z\tau}$ in Eq. (4) with $0 < \tau < \beta$ and $f(z)$ being finite at $|z| \to \infty$. By appropriately choosing/deforming the contour $\mathcal{C}$ show that

$$S = \eta \sum_{m=1}^{N_p} n_\eta(z_m) \mathrm{Res}(f, z_m) e^{z_m \tau} \tag{7}$$

if $f(z)$ only has simple poles at $z = z_m$, $m = 1, \ldots, N_p$ with $z_m \neq i\omega_n\, \forall n, m$. In Eq. (7), $\mathrm{Res}(f, z_m)$ denotes the residue of f at z_m. (2 Pts.)

(d) Use the result from (c) to calculate (both for fermions and bosons)

$$\lim_{\tau\to 0^+} T \sum_{n\in\mathbb{Z}} G_0(i\omega_n, \boldsymbol{k}) e^{i\omega_n \tau}, \tag{8}$$

where $G_0(i\omega, \boldsymbol{k}) = (i\omega - \epsilon_{\boldsymbol{k}})^{-1}$ denotes the single-particle (Matsubara) Green's function of a noninteracting system with dispersion $\epsilon_{\boldsymbol{k}}$. (1 Pt.)

(e) Perform the summation in

$$T \sum_{n\in\mathbb{Z}} G_0(i\omega_n, \boldsymbol{k}) G_0(i\omega_n + i\omega_m, \boldsymbol{k} + \boldsymbol{q}), \tag{9}$$

where ω_n and ω_m are fermionic and bosonic Matsubara frequencies, respectively. (3 Pts.)