Übungen zur Theorie der Kondensierten Materie II SS 16

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Blatt 6

Besprechung 03.06.2016

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1. Matsubara Sums (continued) (8 Points)

In part 2(b) of the previous problem sheet you showed that the sum

$$S(t) = T \sum_{n\in\mathbb{Z}} f(i\omega_n) e^{i\omega_n t}, \tag{1}$$

can be written as a contour integral

$$S(t) = \frac{-\eta}{2\pi i} \oint_{\mathcal{C}} dz\ n_\eta(z) f(z) e^{zt}, \tag{2}$$

where $\mathcal{C}$ consists of two vertical lines enclosing the infinite number of poles of n_η.

(a) Assume now that $f(z)$ is analytic everywhere except on the real axis. By deforming $\mathcal{C}$ show that

$$S = \frac{-\eta}{2\pi i} \int_{-\infty}^{\infty} d\omega\ n_\eta(\omega) \left[f(\omega + i0^+) - f(\omega - i0^+)\right] e^{\omega t} \tag{3}$$

holds. You may assume that $f(z)$ decays away in a suitable form as $z \to \infty$. (3 Pts.)

(b) For a non-interacting electron gas the free energy can be written in terms of the Matsubara Green's function

$$F = -T \sum_{\boldsymbol{k}} \sum_n \log\left[-\mathcal{G}^{-1}_{0,\boldsymbol{k}}(\omega_n)\right] e^{i\omega_n 0^+}. \tag{4}$$

Using part (a), compute the sum over n and show that F is the well-known expression from non-interacting fermion theory. In order to carry out the calculation, you will have use the fact that the logarithm can only be defined with a branch-cut (if one wants to avoid multivalued functions). It is convenient to choose a definition where the logarithm $\log z$ has its branch-cut in $[-\infty, 0]$. (5 Pts.)

2. Coherent states (10 Points)

Consider a harmonic oscillator defined by the Hamiltonian $H = \omega a^\dagger a$. The states of this problem are completely specified by the energy eigenstates $\{|n\rangle\}$, with $H|n\rangle = n\omega$. We want to derive a set of states that are eigenstates not of the Hamiltonian but of the creation and annihilation operators. The latter satisfy the commutation relations $[a, a^\dagger] = 1$.

(a) We want to solve the eigenvalue problem given by

$$a|\phi\rangle = \phi|\phi\rangle, \tag{5}$$

where $|\phi\rangle$ is the eigenstate and ϕ is the corresponding eigenvalue. Before we get to this, prove that the seemingly similar eigenvalue problem

$$a^\dagger|\phi\rangle = \phi|\phi\rangle, \tag{6}$$

cannot have a solution. (1 Pt.)

(b) Now we want to find the eigenstates $|\phi\rangle$ with

$$a|\phi\rangle = \phi|\phi\rangle. \tag{7}$$

To solve this problem we first note that the energy eigenstates $\{|n\rangle\}$ provide a complete basis, thus ϕ must be representable as

$$|\phi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle. \tag{8}$$

Find the c_n by inserting this ansatz into equation (7). Choose a normalization with $c_0 = 1$. The $|\phi\rangle$ are the so-called coherent states, they exist for all complex eigenvalues ϕ. (2 Pts.)

(c) It is possible to obtain $|\phi\rangle$ by acting on the empty state:

$$|\phi\rangle = O|0\rangle \tag{9}$$

What is O? (2 Pts.)

(d) The coherent states are not orthogonal to each other. Show that the overlap between two coherent states $|\phi\rangle$ and $|\psi\rangle$ is given by

$$\langle\psi|\phi\rangle = e^{\psi^*\phi}. \tag{10}$$

(2 Pts.)

(e) The coherent states form a complete basis, i.e. the unit operator can be expressed in terms of these. Show that

$$\mathbb{1} = \int \frac{d\mathrm{Re}\phi \cdot d\mathrm{Im}\phi}{\pi} e^{-\phi^*\phi}|\phi\rangle\langle\phi| \tag{11}$$

(2 Pts.)

(f) Assume we have an operator function $f(a)$ acting on $|\phi\rangle$. What is the result? (1 Pt.)