Übungen zur Theorie der Kondensierten Materie II SS 16

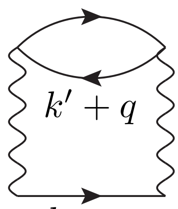
Prof. J. Schmalian
B. Jeevanesan, M. S. Scheurer

Blatt 8
Besprechung 17.06.2016

---

1. Scattering in Fermi liquids (18 Points)

In this exercise we will calculate the diagram



which plays a central role in the microscopic theory of the Fermi liquid as will be discussed both in the lecture and during the tutorial. Here the straight line represents the fermionic single-particle Green's function which we take to be of the form

$$G(i\omega_n, \boldsymbol{k}) = \frac{Z_{\boldsymbol{k}}}{i\omega_n - \epsilon_{\boldsymbol{k}}}, \qquad 0 < Z_{\boldsymbol{k}} \leq 1. \tag{1}$$

For simplicity, you can assume the dispersion to be parabolic, $\epsilon_{\boldsymbol{k}} = \boldsymbol{k}^2/(2m)$, although this is not necessary to solve the following problems. Furthermore, $k = (i\omega_n, \boldsymbol{k})$ and $q = (i\Omega_n, \boldsymbol{q})$ are used to comprise Matsubara frequencies and momenta with ω_n and Ω_n being fermionic and bosonic, respectively. The wiggly lines in the diagram refer to the four-fermion-interaction amplitude $U_{\boldsymbol{q}}$ that in general depends on the transferred momentum $\boldsymbol{q}$.

(a) In order to understand the physical meaning of $Z_{\boldsymbol{k}}$, often referred to as "quasiparticle residue", calculate the spectral function and the occupation number of the single particle state $\boldsymbol{k}$ associated with $G(i\omega_n, \boldsymbol{k})$ in Eq. (1). (1 Pt.)

(b) Returning to our ultimate goal of evaluating the diagram $\Sigma(i\omega_n, \boldsymbol{k})$ shown above, write down its analytical form (in Matsubara formalism) following from the Feynman rules discussed in the lecture. Identify the particle-hole bubble $\Pi(i\Omega_n, \boldsymbol{q})$ that has been a central building block in many calculations of the lecture course. (3 Pts.)

(c) Using the residue theorem with a properly chosen integration contour in the complex plane and subsequent analytic continuation $i\omega_n \to \omega + i0^+$ to the real axis, show that the retarded form $\Sigma^R(\omega, \boldsymbol{k})$ of the diagram is given by

$$\begin{aligned}\Sigma^R(\omega, \boldsymbol{k}) = \int \frac{\mathrm{d}^d\boldsymbol{q}}{(2\pi)^d} U_{\boldsymbol{q}}^2 \Bigg[\mathcal{P}\!\!\int \frac{\mathrm{d}\Omega}{2\pi} \coth\left(\frac{\Omega}{2T}\right) G^R(\omega + \Omega, \boldsymbol{k} + \boldsymbol{q})\, \mathrm{Im}\Pi^R(\Omega, \boldsymbol{q}) \\ + \int \frac{\mathrm{d}\Omega}{2\pi} \tanh\left(\frac{\Omega + \omega}{2T}\right) \Pi^A(\Omega, \boldsymbol{q})\, \mathrm{Im}G^R(\omega + \Omega, \boldsymbol{k} + \boldsymbol{q}) \Bigg]\end{aligned} \tag{2}$$

with d denoting the dimensionality of the system, $\mathcal{P}\!\int$ the principle value integral, G^R/G^A the retarded/advanced Green's function, and $\Pi^R(\Omega, \boldsymbol{q})/\Pi^A(\Omega, \boldsymbol{q})$ the retarded/advanced particle-hole bubble determined by

$$\begin{aligned}\Pi^R(\Omega, \boldsymbol{q}) = 2 \int \frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d} \int \frac{\mathrm{d}\omega}{2\pi} \Bigg[\tanh\left(\frac{\omega}{2T}\right) G^R(\omega + \Omega, \boldsymbol{k} + \boldsymbol{q})\, \mathrm{Im}G^R(\omega, \boldsymbol{k}) \\ + \tanh\left(\frac{\omega + \Omega}{2T}\right) G^A(\omega, \boldsymbol{k})\, \mathrm{Im}G^R(\omega + \Omega, \boldsymbol{k} + \boldsymbol{q}) \Bigg]\end{aligned} \tag{3}$$

and similarly for $\Pi^A(\Omega, \boldsymbol{q})$. (6 Pts.)

(d) Let us first focus on the imaginary part of $\Sigma^R(\omega, \boldsymbol{k})$. Convince yourself that $\Pi^R(\Omega, \boldsymbol{q})$ enters $\mathrm{Im}\Sigma^R(\omega, \boldsymbol{k})$ only in the form of its imaginary part $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$. Focusing on small T and ω (compared to the Fermi energy E_F), which allows neglecting ω and Ω in the delta functions appearing in the expression for $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$ following from Eq. (3), show that

$$\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q}) \sim A_{\boldsymbol{q}}\, \Omega \tag{4}$$

and find the explicit form of the prefactor $A_{\boldsymbol{q}}$. To obtain Eq. (4) you can take the density of states to be independent of the direction normal to the Fermi surface. (3 Pts.)

(e) Using this result, we obtain

$$\mathrm{Im}\Sigma^R(\omega, \boldsymbol{k}) \sim B_{\boldsymbol{k}} \left(\omega^2 + \pi^2 T^2\right) \tag{5}$$

for $\omega, T \ll E_F$. This is the typical behavior of a Fermi liquid. Determine an expression for the prefactor $B_{\boldsymbol{k}}$ in terms of the quasiparticle residues? (3 Pts)

(f) Show that $\mathrm{Re}\Sigma^R(\omega, \boldsymbol{k}) \sim C_{\boldsymbol{k}}\omega$ using Kramers-Kronig relations (a cutoff is required to make the integrals convergent). (1 Pt.)

(g) The concept of quasiparticles is meaningful as long as $\mathrm{Im}\Sigma^R(\Omega, \boldsymbol{q})$ goes faster to zero than ω in the limit $\omega \to 0$. The case $\mathrm{Im}\Sigma^R(\omega) \sim D|\omega|$ is thus sometimes referred to as "marginal Fermi liquid". What is the asymptotic behavior of $\mathrm{Re}\Sigma^R(\omega)$ in a marginal Fermi liquid? (1 Pt)