Übungen zur Theorie der Kondensierten Materie II SS 16

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1. Polarization operator of graphene (18 Points)

In this exercise we will calculate the retarded polarization operator $\Pi^R(\Omega, \boldsymbol{q})$ of a single Dirac cone of graphene, i.e., of a 2D system with Bloch Hamiltonian

$$h_{\boldsymbol{k}} = v_F(\sigma_x k_x + \sigma_y k_y). \tag{1}$$

Here σ_x and σ_y are Pauli matrices in pseudospin space. In the following we will set $v_F = 1$ for notational convenience.

(a) Show that the Matsubara Green's function of the system can be written as

$$G(i\omega_n, \boldsymbol{k}) = \sum_{s=+,-} \frac{\mathcal{P}_{s,\boldsymbol{k}}}{i\omega_n - s|\boldsymbol{k}|}, \qquad \mathcal{P}_{s,\boldsymbol{k}} = \frac{1}{2}\left(1 + s\frac{\boldsymbol{k}\cdot\boldsymbol{\sigma}}{|\boldsymbol{k}|}\right). \tag{2}$$

What is the meaning of the operators $\mathcal{P}_{s,\boldsymbol{k}}$? (2 Pts.)

(b) On the imaginary axis, the polarization operator is given by

$$\Pi(i\Omega_n, \boldsymbol{q}) = T\sum_{\omega_n}\int \frac{\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2}\mathrm{Tr}\left[G(i\omega_n, \boldsymbol{k})G(i\omega_n + i\Omega_n, \boldsymbol{k}+\boldsymbol{q})\right] \tag{3}$$

with Ω_n and ω_n being bosonic and fermionic, respectively. Evaluate the trace Tr[...], which refers to pseudospin space, using the representation (2) of the Green's function and the Matsubara summation. Perform the the analytic continuation $\Pi(i\Omega_n, \boldsymbol{q}) \to \Pi^R(\Omega, \boldsymbol{q})$. (3 Pts.)

(c) For simplicity, we will focus on zero temperature in the following. Identify the relevant terms in the expression for the polarization operator obtained in part (b). (1 Pt.)

(d) To perform the momentum integration, it is convenient to introduce new variables ξ and η via

$$\xi = |\boldsymbol{k}| + |\boldsymbol{k}+\boldsymbol{q}|, \qquad \eta = |\boldsymbol{k}| - |\boldsymbol{k}+\boldsymbol{q}|. \tag{4}$$

What is the geometric meaning of ξ and η? In order to rewrite the momentum integration, show that

$$\cos(\theta_{\boldsymbol{k},\boldsymbol{k}+\boldsymbol{q}}) = \frac{2\boldsymbol{q}^2 - \eta^2 - \xi^2}{\eta^2 - \xi^2} \tag{5}$$

with $\theta_{\boldsymbol{k},\boldsymbol{k}+\boldsymbol{q}}$ denoting the angle between $\boldsymbol{k}$ and $\boldsymbol{k}+\boldsymbol{q}$ and that

$$\mathrm{d}k_x\mathrm{d}k_y = \frac{\xi^2 - \eta^2}{4\sqrt{(\boldsymbol{q}^2 - \eta^2)(\xi^2 - \boldsymbol{q}^2)}}\mathrm{d}\xi\mathrm{d}\eta. \tag{6}$$

You might find it useful to choose $\boldsymbol{q} = (q_x, 0)$ which is possible without loss of generality. (3 Pts.)

(e) Rewrite the momentum integral in terms of ξ and η and perform the integrations using the identity

$$\int_1^\infty \mathrm{d}x \frac{x}{\sqrt{x^2-1}} \frac{1}{w^2-x^2} = \frac{-\pi}{2\sqrt{1-w^2}} \tag{7}$$

for $w \in \mathbb{C}$ (except for $w \in \mathbb{R}$ with $|w| \geq 1$). (3 Pts.)

(f) Sketch $\mathrm{Re}\Pi^R(\Omega, \boldsymbol{q})$ and $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$ as a function of Ω. (3 Pts.)

(g) Check your result for $\mathrm{Im}\Pi^R(\Omega, \boldsymbol{q})$ by using $\frac{1}{(x+i0^+)} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$ before performing the ξ integration. (3 Pts.)