Karlsruher Institut für Technologie | Institut für Theorie der Kondensierten Materie

Übungen zur Theorie der Kondensierten Materie II SS 16

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1. DC Josephson effect (9 Points)

In this exercise, we will derive the DC Josephson effect microscopically. For this purpose, we consider two identical superconductors described by the BCS mean-field Hamiltonians ($\Delta \in \mathbb{R}$)

$$H_c = \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \left(\Delta e^{-i\varphi_c} \sum_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} + \text{H.c.} \right), \tag{1a}$$

$$H_d = \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} d^\dagger_{\boldsymbol{k}\sigma} d_{\boldsymbol{k}\sigma} - \left(\Delta e^{-i\varphi_d} \sum_{\boldsymbol{k}} d^\dagger_{\boldsymbol{k}\uparrow} d^\dagger_{-\boldsymbol{k}\downarrow} + \text{H.c.} \right), \tag{1b}$$

which are coupled by the single-electron-tunneling operator

$$H_t = \sum_{\boldsymbol{k},\boldsymbol{k}',\sigma} t\, c^\dagger_{\boldsymbol{k}\sigma} d_{\boldsymbol{k}'\sigma} + \text{H.c.}\,. \tag{2}$$

The total Hamiltonian of the system is given by $H_c + H_d + H_t$.

(a) Choose a gauge that removes the phases φ_c and φ_d from Eq. (1). How does the tunneling amplitude t change? (1 Pt.)

(b) Show that the Josephson current I_J is given by

$$I_J = 2e \left\langle \partial_{\Delta\varphi} H_t \right\rangle, \qquad \Delta\varphi = \varphi_c - \varphi_d, \tag{3}$$

where e denotes the elementary charge. (2 Pts.)

(c) Treating H_t as a small perturbation to $H_c + H_d$, expand Eq. (3) to leading nontrivial order in H_t. Perform the contractions using Wick's theorem and bring the result into the form

$$I_J = I_c \sin(\Delta\varphi). \tag{4}$$

Express the critical current I_c in terms of the Nambu Green's functions of a BCS superconductor. (4 Pts.)

(d) Evaluate the Matsubara and momentum sums to find the explicit form of I_c. You can take the normal state density of states to be independent of energy. (2 Pts.)

2. Spin-susceptibility of a superconductor (9 Points)

Here we consider an isolated BCS superconductor and calculate its dynamic spin-susceptibility

$$\chi_{jj'}(i\Omega_n, \boldsymbol{q}) = \int_0^\beta \mathrm{d}\tau \, \langle T_\tau M_j(\boldsymbol{q},\tau) M_{j'}(-\boldsymbol{q},0)\rangle \, e^{i\Omega_n \tau}, \qquad M_j(\boldsymbol{q},\tau) = \sum_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}}(\tau)\sigma_j c_{\boldsymbol{k}+\boldsymbol{q}}(\tau) \tag{5}$$

with $j, j' = 1, 2, 3$, $c_{\boldsymbol{k}} = (c_{\boldsymbol{k}\uparrow}, c_{\boldsymbol{k}\downarrow})^T$, σ_j denoting Pauli matrices, and Ω_n being bosonic. We first work on the imaginary axis and later perform the analytic continuation.

(a) In a system with spin-rotation symmetry (no spin-orbit coupling), it holds $\chi_{jj'}(i\Omega_n, \boldsymbol{q}) = \delta_{jj'}\chi_{33}(i\Omega_n, \boldsymbol{q})$. Why? (1 Pt.)

(b) Express M_3 in terms of Nambu spinors. Use this to show that the susceptibility can be represented by a bubble diagram where the fermionic lines refer to Nambu Green's functions. (3 Pts.)

(c) Perform the Matsubara summation to show that

$$\chi_{jj'}(i\Omega_n, \boldsymbol{q}) = \frac{1}{2}\delta_{jj'} \sum_{\boldsymbol{k}} \sum_{\lambda,\lambda'=+,-} \left[1 + \frac{\epsilon_{\boldsymbol{k}}\epsilon_{\boldsymbol{k}+\boldsymbol{q}} + \Delta^2}{\lambda\lambda' E_{\boldsymbol{k}} E_{\boldsymbol{k}+\boldsymbol{q}}}\right] \frac{n_F(\lambda' E_{\boldsymbol{k}+\boldsymbol{q}}) - n_F(\lambda E_{\boldsymbol{k}})}{i\Omega_n - (\lambda' E_{\boldsymbol{k}+\boldsymbol{q}} - \lambda E_{\boldsymbol{k}})}, \tag{6}$$

where $E_{\boldsymbol{k}} = \sqrt{\epsilon_{\boldsymbol{k}}^2 + \Delta^2}$ and n_F is the Fermi function. (2 Pts.)

(d) For example, the NMR relaxation rate is determined by

$$\frac{1}{T_1 T} \propto \sum_{\boldsymbol{q}} \lim_{\Omega\to 0} \frac{\mathrm{Im}\chi^R_{jj}(\Omega, \boldsymbol{q})}{\Omega}. \tag{7}$$

Using the expression (6) of the susceptibility, the analytic continuation to the retarded function $\chi^R_{jj}(\Omega, \boldsymbol{q})$ is straightforward. Write the right hand side of Eq. (7) as a single integral over the quasi-particle energies E of the superconductor (the normal state density of states can be approximated as a constant). (3 Pts.)