Übungen zur Theorie der Kondensierten Materie II   SS 18

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Blatt 6

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1. Large N limit of the φ^4 model (5 + 10 + 10 + 15 + 35 + 25 Points)

You are by now familiar with the φ^4 theory. In this exercise we want to consider the so called large N limit of the theory. N is the number of components φ_i, $i \in \{1, 2, ..., N\}$, of the field vector $\vec{\varphi}$. The action is then written in the Form

$$S = \frac{1}{N} \int d^d x \left(\frac{r_0}{2} \vec{\varphi}(x) \cdot \vec{\varphi}(x) + \frac{1}{2} (\nabla \vec{\varphi}(x)) \cdot (\nabla \vec{\varphi}(x)) - \frac{u_0}{4N} (\vec{\varphi}(x) \cdot \vec{\varphi}(x))^2 \right),$$

where the dot product $\cdot$ is the scalar product in the N components.

(a) Show that the free Green's function of the action for $u_0 = 0$ is

$$G^0_{ij}(r_0, q) = \langle \varphi_i(q) \varphi_j(-q) \rangle = \frac{\delta_{ij}}{r_0 + q^2},$$

where $i, j \in \{1, 2, ..., N\}$.

(b) Consider the two second order diagrams shown in Fig. 1. What is their respective order in N? Which diagram is more important at large N?

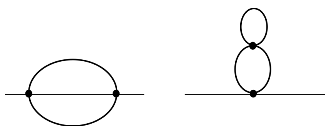

Abbildung 1: Two second order graphs.

(c) Argue that the full Green's function $G(r, q)$ at large N can be graphically represented by the Dyson equation Fig. 2.

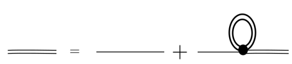

Abbildung 2: The Dyson equation.

(d) The Dyson equation reads

$$G(q) = G^0(q) + G^0(q) \left[\int^{\Lambda} \frac{d^d q'}{(2\pi)^d} G(q') \right] G(q),$$

where Λ is the cut-off, i.e. the radial integration runs from $q' = 0$ to $q' = \Lambda$. Show that from this equation follows that $G(q) = 1/(r + q^2)$ where the following expression for r can be obtained:

$$r = r_0 + u \int_{q'}^{\Lambda} G(r, q') . \tag{1}$$

Here, we abbreviated $\int^{\Lambda} \frac{d^d q'}{(2\pi)^d}$ by $\int_q^{\Lambda}$. Notice, that the full Green's function is obtained from the free one by substituting $r_0 \to r$.

(e) Now, derive the renormalization group equations for r and u from Eq. (1). Show that the integration in (1) can be splitted as

$$r = r_0 + u \int_{q'}^{\Lambda/b} G(r, q') + u K_d \Lambda^d l G(r, \Lambda)$$

for a small l and $b = e^l$. Show that this expression can equivalently be written

$$r = r_0 + u b^{2-d} \int_{q'}^{\Lambda} G\left(b^2 r, q'\right) + u K_d \Lambda^d l G(r, \Lambda) . \tag{2}$$

Note, that it holds $G(r, b^{-1} q) = b^2 G(b^2 r, q) = b^2 G(r', q)$. K_d is a numerical cooefficient that accounts for the angular part of the integral. This procedure is called a scale transformation. Approximating $G(r, \Lambda) \approx G(0, \Lambda) + \partial_r \, G(r, \Lambda)|_{r=0} \, r$, derive the expression

$$r' = b^2 r = \frac{b^2 r_0 + u K_d \Lambda^d l G(0, \Lambda) + u b^{4-d} \int_q^{\Lambda} G\left(b^2 r, q\right)}{1 - u K_d \Lambda^d l \partial_r \, G(r, \Lambda)|_{r=0}} .$$

Expand the above expression to second order in the small u. Compare it to the equation (1) after the scale transformation:

$$r' = r_0' + u' \int_{q'}^{\Lambda} G(r, q') .$$

Find r_0' and u' to first order in l.

(f) The derived r_0', u' are the expressions for r and u after the scale transformation. Consider the case $d = 4 - \varepsilon$. Derive differential equations of the form

$$\begin{aligned} \frac{dr_0}{dl} &= f(r_0, u) \\ \frac{du}{dl} &= g(r_0, u) , \end{aligned}$$

that describe how an infinitesimal scale transformation (infinitesimal in the sense that l is infinitessimaly small) changes r_0 and u. These equations are called flow equations. Determine the so called fix-point values of u at which holds

$$\frac{du}{dl} = 0.$$

How many fixed points do exist. Which ones are stable and unstable?