Karlsruher Institut für Technologie

Institut für Theorie der Kondensierten Materie

**Übungen zur Theorie der Kondensierten Materie II SS 18**

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**Blatt 3**

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**1. Density-density response in one dimension** (30 + 20 Points)

In this problem we calculate the Lindhard function in one spatial dimension. The density-density response of a free-electron gas is given in momentum space by the relation

$$\chi_q^r(\omega) = \frac{1}{L}\sum_k f_k \left( \frac{1}{\omega + i0^+ - \epsilon_k + \epsilon_{k-q}} - \frac{1}{\omega + i0^+ + \epsilon_k - \epsilon_{k+q}} \right), \tag{1}$$

where $f_k$ is the Fermi-Dirac distribution and $L$ is the length of the system. The energies $\epsilon_k$ are the energies of free particles

$$\epsilon_k = \frac{k^2}{2m} - \mu. \tag{2}$$

(a) Replace the sum in (1) by an integral and determine Re $\left[\chi_q^r(\omega)\right]$ and Im $\left[\chi_q^r(\omega)\right]$ in one dimension for $T = 0$.

(b) The real part of the dielectric function $\varepsilon(q,\omega) = 1 - V(q)\,\chi_q^r(\omega)$ vanishes for certain pairs of values of $(q,\omega)$. This phenomenon is called plasmon resonance. Using the connection between the dielectric function and the density-density response, find the dispersion relation $\omega(q)$ of plasmons. Then, assume that $V(q) = V$, i.e. that we are dealing with point interactions. Expand the relation $\omega(q)$ for small $q$. Is Im$(\varepsilon)$ finite for $\omega(q)$? What is the physical meaning of the imaginary part of $\varepsilon$?

**2. Orthogonality catastrophe I** (50 Points)

The fact that the many particle ground states of a non-interacting fermi gas $|\psi\rangle$ without a local perturbation and $|\psi'\rangle$ with a local perturbation are orthogonal in the thermodynamic limit is named orthogonality catastrophe:

$$|\langle\psi|\psi'\rangle|^2 \xrightarrow{N\to\infty} 0\ ,$$

where $N$ is the particle number (P. W. Anderson, Phys. Rev. Lett. **18**, 1049 (1967)). This implies that it takes such a perturbation a very long time to decay (by Fermi's golden rule). In this exercise, we want to understand the phenomenon qualitatively. This is the first part of the exercise. It will be continued next week.

(a) Show that $|\langle\psi|\psi'\rangle|^2$ is bounded from above in the following manner:

$$|\langle\psi|\psi'\rangle|^2 < \exp -\frac{1}{2}\sum_{\epsilon_n \le E_F,\, e_{m'} > E_F} |\langle n|m'\rangle|^2,$$

where $E_F$ is the Fermi energy and $|n\rangle$, $|m'\rangle$ are the single particle states of the unperturbed and the perturbed gas. Possible procedure: use the fact that $|\psi\rangle$ and $|\psi'\rangle$ can be written as slater determinants and that the scalar product of two slater determinants is given by the determinant of the overlap matrix:

$$\langle\psi|\psi'\rangle = \det A,$$

where $A_{nm} = \langle n|m'\rangle$ (show it, if you feel like showing it). Since only energy levels below the Fermi level are occupied, we know that $A_{nm} \neq 0$ for $\epsilon_n, \epsilon_m < E_F$.
For a matrix $\tilde{A}$ with unit vectors as rows holds $\det\tilde{A} \leq 1$. Express $A$ through $\tilde{A}$ and obtain a bound on $\ln|\langle\psi|\psi'\rangle|$:

$$\ln|\langle\psi|\psi'\rangle| \leq \frac{1}{2}\sum_n \ln\left[\sum_{m'} |\langle n|m'\rangle|^2\right].$$

Now, insert a non-trivial "one" into the above expression, and, assuming that the perturbation is small and therefore $\sum_{\epsilon_{m'}>E_F} |\langle n|m'\rangle|^2 \ll 1$, derive

$$\ln|\langle\psi|\psi'\rangle|^2 \leq -\sum_{\epsilon_n\leq E_F, e_{m'}>E_F} |\langle n|m'\rangle|^2 .$$

Next week, we will see that the sum $\sum_{\epsilon_n\leq E_F, e_{m'}>E_F} |\langle n|m'\rangle|^2 \sim \ln N$ and therefore the overlap $|\langle\psi|\psi'\rangle|^2$ vanishes in the thermodynamic limit.