Karlsruher Institut für Technologie | Institut für Theorie der Kondensierten Materie

Übungen zur Theorie der Kondensierten Materie II SS 18

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1. Renormalization of the φ^3 theory II (20 points)

(a) Restate the main steps of the RG procedure in your own words and equations. Use the example of the φ^4 theory.

2. Matsubara summation (10 + 5 + 10 + 5 +20 Points)

In this exercise, we will learn how sums of the form

$$S = T \sum_{n\in\mathbb{Z}} h(i\omega_n) \tag{1}$$

with $\omega_n = 2n\pi T$ for bosons ($\eta = -1$ in the following) and $\omega_n = (2n+1)\pi T$ for fermions ($\eta = +1$) can be very efficiently evaluated at arbitrary temperature T. This is important as we will encounter expressions of the form (1) very frequently in the remainder of the lecture course.

(a) As a first step, determine the poles of the Fermi ($\eta = +1$) and Bose ($\eta = -1$) function,

$$n_\eta(z) = \frac{1}{e^{\beta z} + \eta}, \qquad \beta = T^{-1}, \tag{2}$$

and the associated residues.

(b) With this in mind, show that one can write

$$S = \frac{-\eta}{2\pi i} \oint_{\mathcal{C}} \mathrm{d}z\, n_\eta(z) h(z), \tag{3}$$

where the contour $\mathcal{C}$ encloses the infinite set of points $\{i\omega_n | n \in \mathbb{Z}\}$ in a counter-clockwise manner and $h(z)$ is analytic in the domain bound by $\mathcal{C}$.

(c) As a first example, let $h(z) = f(z)e^{z\tau}$ in Eq. (1) with $0 < \tau < \beta$ and $f(z)$ being finite at $|z| \to \infty$. By appropriately choosing/deforming the contour $\mathcal{C}$ show that

$$S = \eta \sum_{m=1}^{N_p} n_\eta(z_m) \mathrm{Res}(f, z_m) e^{z_m \tau} \tag{4}$$

if $f(z)$ only has simple poles at $z = z_m$, $m = 1, \ldots, N_p$ with $z_m \neq i\omega_n \, \forall n, m$. In Eq. (4), $\mathrm{Res}(f, z_m)$ denotes the residue of f at z_m.

(d) Use the result from (c) to calculate (both for fermions and bosons)

$$\lim_{\tau \to 0^+} T \sum_{n \in \mathbb{Z}} G_0(i\omega_n, \boldsymbol{k}) e^{i\omega_n \tau}, \tag{5}$$

where $G_0(i\omega, \boldsymbol{k}) = (i\omega - \epsilon_{\boldsymbol{k}})^{-1}$ denotes the single-particle (Matsubara) Green's function of a noninteracting system with dispersion $\epsilon_{\boldsymbol{k}}$.

(e) Perform the summation in

$$T \sum_{n \in \mathbb{Z}} G_0(i\omega_n, \boldsymbol{k}) G_0(i\omega_n + i\omega_m, \boldsymbol{k} + \boldsymbol{q}), \tag{6}$$

where ω_n and ω_m are fermionic and bosonic Matsubara frequencies, respectively.