**Übungen zur Theoretischen Physik F SS 09**

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## 1. Ising-Modell

(a)

$$Z(N) = \sum_{\{\sigma_i^z\}} \exp\left[\frac{J}{k_B T}\sum_{i=1}^{N-1}\sigma_i^z\sigma_{i+1}^z\right] = \sum_{\{\sigma_i^z\}}\prod_{i=1}^{N-1}\exp\left[\frac{J}{k_B T}\sigma_i^z\sigma_{i+1}^z\right],$$

$$\exp\left[\frac{J}{k_B T}\sigma_i^z\sigma_{i+1}^z\right] = \cosh\frac{J}{k_B T} + \sigma_i^z\sigma_{i+1}^z\sinh\frac{J}{k_B T},$$

$$Z(N) = 2^N\left[\cosh\frac{J}{k_B T}\right]^{N-1}.$$

oder

$$\begin{aligned} Z(3) &= \sum_{\{\sigma_i^z\}}\prod_{i=1}^{2}\left(\cosh\frac{J}{k_B T} + \sigma_i^z\sigma_{i+1}^z\sinh\frac{J}{k_B T}\right) \qquad (1)\\ &= \sum_{\{\sigma_i^z\}}\left(\cosh^2\frac{J}{k_B T} + (\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z)\cosh\frac{J}{k_B T}\sinh\frac{J}{k_B T} + \sigma_1^z\sigma_2^z\sigma_2^z\sigma_3^z\sinh^2\frac{J}{k_B T}\right)\\ &= 2^3\cosh^2\frac{J}{k_B T}, \end{aligned}$$

weil

$$\sum_{\{\sigma_i^z\}}\sigma_1^z\sigma_2^z = \sum_{\{\sigma_i^z\}}\sigma_2^z\sigma_3^z = 0,$$

und

$$\sum_{\{\sigma_i^z\}}\sigma_1^z\sigma_2^z\sigma_2^z\sigma_3^z = \sum_{\{\sigma_i^z\}}\sigma_1^z\sigma_3^z = 0.$$

(b)

$$F = -k_B T\ln Z(3) = -3k_B T\ln 2 - 2k_B T\ln\cosh\frac{J}{k_B T}.$$

(c)

$$S = -\frac{\partial F}{\partial T} = 3k_B\ln 2 + 2k_B\ln\cosh\frac{J}{k_B T} - 2\frac{J}{T}\tanh\frac{J}{k_B T},$$

$$c_H = T\left(\frac{\partial S}{\partial T}\right)_H = \frac{J^2}{k_B T^2}\frac{1}{\cosh^2[J/(k_B T)]}.$$

(d)

$$\langle \sigma_i^z \rangle = \frac{1}{Z(3)} \sum_{\{\sigma_i^z\}} \sigma_i^z \exp\left[\frac{J}{k_B T} \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z\right] = 0,$$

weil alle Glieder antisymmetrisch sind.

(e)

$$M(H) = \frac{1}{Z(H,N)} \sum_{\{\sigma_i^z\}} \sum_{j=1}^{N} \mu \sigma_j^z \exp\left[\frac{J}{k_B T} \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z + \frac{\mu H}{k_b T} \sum_{i=1}^{N} \sigma_i^z\right],$$

wobei

$$Z(H,N) = \sum_{\{\sigma_i^z\}} \exp\left[\frac{J}{k_B T} \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z + \frac{\mu H}{k_b T} \sum_{i=1}^{N} \sigma_i^z\right].$$

Wenn $\mu H \ll k_B T$,

$$\exp\left[\frac{\mu H}{k_B T} \sigma_i^z\right] \approx 1 + \frac{\mu H}{k_B T} \sigma_i^z,$$

und $Z(H,N) \approx Z(N)$.

Dann

$$M = \chi H,$$

mit

$$\chi = \frac{\mu^2}{k_B T} \sum_{i=1}^{N} \sum_{j=1}^{N} \langle \sigma_i^z \sigma_j^z \rangle.$$

$$\langle \sigma_i^z \sigma_j^z \rangle = \frac{1}{Z} \sum_{\{\sigma_i^z\}} \sigma_i^z \sigma_j^z \exp\left[\frac{J}{k_B T} \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z\right] = \left[\tanh \frac{J}{k_B T}\right]^{|i-j|}.$$

Für $N = 3$ [sehen Sie (1)]:

$$\langle (\sigma_i^z)^2 \rangle = 1,$$

$$\langle \sigma_1^z \sigma_2^z \rangle = \langle \sigma_2^z \sigma_3^z \rangle = \frac{1}{Z(3)} 2^3 \cosh \frac{J}{k_B T} \sinh \frac{J}{k_B T} = \tanh \frac{J}{k_B T},$$

$$\langle \sigma_1^z \sigma_3^z \rangle = \frac{1}{Z(3)} 2^3 \sinh^2 \frac{J}{k_B T} = \tanh^2 \frac{J}{k_B T}.$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \langle \sigma_i^z \sigma_j^z \rangle = 3 + 2\left(\langle \sigma_1^z \sigma_2^z \rangle + \langle \sigma_2^z \sigma_3^z \rangle + \langle \sigma_1^z \sigma_3^z \rangle\right) = 3 + 2\left(2 \tanh \frac{J}{k_B T} + \tanh^2 \frac{J}{k_B T}\right).$$

$$M = \frac{\mu^2 H}{k_B T}\left[1 + 2\left(1 + \tanh \frac{J}{k_B T}\right)^2\right].$$

(f)

$$\chi = \frac{\mu^2}{k_B T}\left[1 + 2\left(1 + \tanh \frac{J}{k_B T}\right)^2\right].$$

**2. Harmonische Oszillator:**

**(a)**

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

$$Z = \int dxdp\ e^{-\beta H} = \underbrace{\int dp\ e^{-\frac{\beta}{2m}p^2}}_{\left(\frac{2\pi m}{\beta}\right)^{1/2}} \underbrace{\int dx\ e^{-\beta\frac{m}{2}\omega^2 x^2}}_{\left(\frac{2\pi}{\beta m\omega^2}\right)^{1/2}}$$

$$\Rightarrow \boxed{Z = \frac{2\pi k_B T}{\omega}}$$

$$F = -k_B T \ln Z \quad \Rightarrow \quad \boxed{F = -k_B T \ln \frac{2\pi k_B T}{\omega}}$$

$$S = -\frac{\partial F}{\partial T} \quad \Rightarrow \quad \boxed{S = k_B \ln \frac{2\pi k_B T}{\omega} + k_B}$$

$$U = F + TS \quad \Rightarrow \quad \boxed{U = k_B T}$$

$$c_V = \left(\frac{\partial U}{\partial T}\right)_V \quad \Rightarrow \quad \boxed{c_V = k_B}$$

**(b)**

$$H = \hbar\omega\left(a^+ a + \frac{1}{2}\right)$$

$$Z = Z_1 = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)} = e^{-\beta\hbar\omega/2}\frac{1}{1 - e^{-\beta\hbar\omega}} = \boxed{\left[2\sinh\left(\frac{\beta\hbar\omega}{2}\right)\right]^{-1} = Z}$$

$$F = -k_B T \ln Z \quad \Rightarrow \quad \boxed{F = k_B T \ln\left[2\sinh\left(\frac{\hbar\omega}{2k_B T}\right)\right]}$$

$$S = -\frac{\partial F}{\partial T} \quad \Rightarrow \quad \boxed{S = -k_B \ln\left[2\sinh\left(\frac{\hbar\omega}{2k_B T}\right)\right] + \frac{\hbar\omega}{2T}\coth\left(\frac{\hbar\omega}{2k_B T}\right)}$$

$$U = F + TS \quad \Rightarrow \quad \boxed{U = \frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2k_B T}\right)}$$

$$c_V = \left(\frac{\partial U}{\partial T}\right)_V \quad \Rightarrow \quad \boxed{c_V = \frac{1}{k_B}\left(\frac{\hbar\omega}{2T}\right)^2 \frac{1}{\sinh^2\left(\frac{\hbar\omega}{2k_BT}\right)}}$$

$$T \to \infty \quad : \quad U = k_BT, \quad c_V = k_B \quad \text{wie klassisch}$$

$$T \to 0 \quad : \quad U = \frac{\hbar\omega}{2} \quad \text{Nullpunktsenergie}$$

$$c_V = \frac{1}{k_B}\left(\frac{\hbar\omega}{2T}\right)^2 e^{-\frac{\hbar\omega}{k_BT}} \propto \frac{\Delta^2}{T^2} e^{-\Delta/k_BT}, \quad \Delta = \hbar\omega = \text{Energielücke}$$