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Übungen zur Theoretischen Physik F SS 11

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1. Zustandsdichte in niedrigen Dimensionen:

$$\frac{1}{V}\sum_{\mathbf{k}} \to \int \frac{d^2k}{(2\pi\hbar)^2},$$

$$\epsilon(\mathbf{k}) = \frac{k^2}{2m},$$

(a) Zwei Dimensionen:

$$\nu(\epsilon) = \int \frac{d^2k}{(2\pi\hbar)^2}\,\delta\left(\epsilon - \frac{k^2}{2m}\right) = \frac{1}{2\pi\hbar^2}\int_0^\infty kdk\,\delta\left(\epsilon - \frac{k^2}{2m}\right)$$

$$= \frac{m}{2\pi\hbar^2}\int_0^\infty d\left(\frac{k^2}{2m}\right)\delta\left(\epsilon - \frac{k^2}{2m}\right) = \frac{m}{2\pi\hbar^2},$$

$$\boxed{\nu_{2D}(\epsilon) = \frac{m}{2\pi\hbar^2}}$$

(b) Ein Dimension:

$$\nu(\epsilon) = \int_{-\infty}^{\infty} \frac{dk}{2\pi\hbar}\,\delta\left(\epsilon - \frac{k^2}{2m}\right) = \frac{2m}{\pi\hbar}\int_0^\infty dk\,\delta\left(2m\epsilon - k^2\right)$$

$$= \frac{m}{\pi\hbar}\int_0^\infty \frac{d(k^2)}{k}\,\delta\left(2m\epsilon - k^2\right) = \frac{1}{\pi\hbar}\sqrt{\frac{m}{2\epsilon}},$$

$$\boxed{\nu_{1D}(\epsilon) = \frac{1}{\pi\hbar}\sqrt{\frac{m}{2\epsilon}}}$$

2. Besetsungszahlen in einem idealen Fermi-Gas:

In einem Fermi-Gas

$$n_\lambda = \begin{cases} 1, & \text{wenn } \lambda \text{ besetzt ist} \\ 0, & \text{wenn } \lambda \text{ leer ist} \end{cases} \Rightarrow\ n_\lambda = n_\lambda^2.$$

Die Teilchen sind ununterscheidbar und *unabhängig*.

Deswegen (hier $W_\lambda(n_\lambda)$ ist die Warscheinlichkeit dass der Zustand λ besetzt ist)

(a)

$$\langle n_\lambda^2 \rangle = \sum_{n_\lambda} n_\lambda^2 W_\lambda(n_\lambda) = \sum_{n_\lambda} n_\lambda W_\lambda(n_\lambda) = \langle n_\lambda \rangle$$

(b) Wenn $\lambda_1 \neq \lambda_2$:

$$W_{\lambda_1\lambda_2}(n_{\lambda_1\lambda_2}) = W_{\lambda_1}(n_{\lambda_1}) W_{\lambda_2}(n_{\lambda_2}),$$

$$\langle n_{\lambda_1} n_{\lambda_2} \rangle = \sum_{n_{\lambda_1} n_{\lambda_2}} n_{\lambda_1} n_{\lambda_2} W_{\lambda_1\lambda_2}(n_{\lambda_1\lambda_2}) = \sum_{n_{\lambda_1} n_{\lambda_2}} n_{\lambda_1} n_{\lambda_2} W_{\lambda_1}(n_{\lambda_1}) W_{\lambda_2}(n_{\lambda_2})$$
$$= \sum_{n_{\lambda_1}} n_{\lambda_1} W_{\lambda_1}(n_{\lambda_1}) \sum_{n_{\lambda_2}} n_{\lambda_2} W_{\lambda_2}(n_{\lambda_2}) = \langle n_{\lambda_1} \rangle \langle n_{\lambda_2} \rangle.$$

(c)

$$\langle N \rangle = \sum_\lambda \langle n_\lambda \rangle.$$

$$\langle N \rangle^2 = \left(\sum_\lambda \langle n_\lambda \rangle \right)^2 = \sum_\lambda \langle n_\lambda \rangle^2 + \sum_{\lambda_1 \neq \lambda_2} \langle n_{\lambda_1} \rangle \langle n_{\lambda_2} \rangle$$

$$\langle N^2 \rangle = \left\langle \left(\sum_\lambda n_\lambda \right)^2 \right\rangle = \sum_\lambda \langle n_\lambda^2 \rangle + \sum_{\lambda_1 \neq \lambda_2} \langle n_{\lambda_1} n_{\lambda_2} \rangle$$

$$(\Delta N)^2 \equiv \langle N^2 \rangle - \langle N \rangle^2 = \sum_\lambda \left[\langle n_\lambda^2 \rangle - \langle n_\lambda \rangle^2 \right] = \sum_\lambda \langle n_\lambda \rangle \left[1 - \langle n_\lambda \rangle \right],$$

deswegen

$$(\Delta N)^2 = \langle N \rangle - \sum_\lambda \langle n_\lambda \rangle^2 \leqslant \langle N \rangle$$

Letztendlich

$$\Delta N \leqslant \sqrt{\langle N \rangle}\ \Rightarrow\ \frac{\Delta N}{\langle N \rangle} \leqslant \frac{1}{\sqrt{\langle N \rangle}}.$$

Wenn $T = 0$, dann $\langle n_\lambda \rangle = 0, 1$, deswegen

$$\Delta N(T = 0) = 0.$$

3. Thermodynamik des idealen Fermi-Gases:

$$\sum_\lambda \to \sum_\sigma V \int \frac{d^3p}{(2\pi\hbar)^3} = (2s+1)V \int d\epsilon \nu(\epsilon) \int \frac{d\Omega}{4\pi},$$

wobei $d\Omega = \sin\theta d\theta d\varphi$.

$$\nu(\epsilon) = \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\sqrt{\epsilon}.$$

Wenn $T = 0$

$$\mu = E_F = \frac{p_F^2}{2m}; \quad \langle n_\lambda \rangle = \begin{cases} 1, & \epsilon_\lambda \leqslant E_F \\ 0, & \epsilon_\lambda > E_F. \end{cases}$$

(a)

$$N = \sum_\lambda \langle n_\lambda \rangle = (2s+1)V \int_0^{E_F} d\epsilon \nu(\epsilon) = (2s+1)V \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^{E_F} d\epsilon \sqrt{\epsilon} = (2s+1)V \frac{p_F^3}{6\pi^2\hbar^3}$$

(b)

$$U(T=0) = \sum_\lambda \langle n_\lambda \rangle \epsilon_\lambda = (2s+1)V \int_0^{E_F} d\epsilon \nu(\epsilon)\epsilon = (2s+1)V \frac{p_F^5}{20m\pi^2\hbar^3}$$

(c)

$$\Omega = -T \sum_\lambda \ln\left(1 + \exp\frac{\mu - \epsilon_\lambda}{k_B T}\right) = -T(2s+1)V \int d\epsilon \nu(\epsilon) \ln\left(1 + \exp\frac{\mu - \epsilon}{k_B T}\right)$$

$$= -(2s+1)V \int_0^\infty d\epsilon n_F(\epsilon) a(\epsilon),$$

wobei

$$a(\epsilon) = \int_0^\epsilon d\epsilon' \nu(\epsilon') = \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\epsilon d\epsilon' \sqrt{\epsilon'} = \frac{2}{3} \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \epsilon^{3/2}$$

und

$$n_F(\epsilon) = \frac{1}{1 + \exp[\beta(\epsilon - \mu)]}.$$

Deswegen

$$\Omega(T=0) = -(2s+1)V \frac{2}{3} \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^{E_F} d\epsilon \epsilon^{3/2} = -(2s+1)V \frac{p_F^5}{30\pi^2\hbar^3 m}$$

(d)

$$\Omega(T=0) = -\frac{1}{30} \frac{(2s+1)V}{\pi^2\hbar^3} \frac{p_F^5}{m}$$

$$U(T=0) = \frac{1}{20} \frac{(2s+1)V}{\pi^2\hbar^3} \frac{p_F^5}{m}$$

$$\mu N = \frac{1}{12}\frac{(2s+1)V}{\pi^2\hbar^3}\frac{p_F^5}{m}$$

und die Relation

$$\Omega = U - TS - \mu N$$

gilt weil

$$-\frac{1}{30} = \frac{1}{20} - \frac{1}{12}$$

(e)

$$P(T=0) = (2s+1)\frac{p_F^5}{30\pi^2\hbar^3 m} = \frac{1}{5m}\left(\frac{2s+1}{6\pi^2\hbar^3}\right)^{-2/3}\left(\frac{N}{V}\right)^{5/3}$$