Karlsruher Institut für Technologie

Institut für Theorie
der Kondensierten Materie

Übungen zur Theoretischen Physik F SS 11

Prof. Dr. A. Shnirman
Dr. B. Narozhny

Lösungsvorschlag zu Blatt 7
3.6.2011

---

1. Ideales Fermi-Gas:

(a) Das großkanonische Potential:

$$\Omega = -T\sum_\lambda \ln\left(1+e^{\beta(\mu-\epsilon_\lambda)}\right) = -gTV\int\frac{d^3k}{(2\pi\hbar)^3}\ln\left(1+e^{\beta(\mu-\epsilon_{\mathbf{k}})}\right),$$

wobei

$$g = 2s+1 = 2,$$

und

$$\epsilon_{\mathbf{k}} = \frac{k^2}{2m}.$$

Die Zustandsdichte:

$$\int\frac{d^3k}{(2\pi\hbar)^3} = \int d\epsilon\,\nu(\epsilon)\int\frac{d\Omega}{4\pi},\quad \nu(\epsilon) = \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\sqrt{\epsilon}.$$

Deswegen

$$\Omega = -gTV\frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int\limits_0^\infty d\epsilon\sqrt{\epsilon}\ln\left(1+e^{\beta(\mu-\epsilon)}\right).$$

Partielle Integration liefert

$$\Omega = -\frac{2}{3}gV\frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int\limits_0^\infty d\epsilon\; n_F(\epsilon)\;\epsilon^{3/2};\quad n_F(\epsilon) = \frac{1}{1+e^{\beta(\epsilon-\mu)}}.$$

(b) Die Entropie

$$S = -\left(\frac{\partial\Omega}{\partial T}\right)_{V,\mu} = \frac{2}{3}gV\frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int\limits_0^\infty d\epsilon\;\epsilon^{3/2}\frac{\partial}{\partial T}n_F(\epsilon).$$

Wir bemerken, dass

$$\frac{\partial}{\partial T}n_F(\epsilon) = -\frac{\epsilon-\mu}{T}\frac{\partial}{\partial\epsilon}n_F(\epsilon).$$

Deswegen

$$S = -\frac{2}{3}\frac{gV}{T}\frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int\limits_0^\infty d\epsilon\;\epsilon^{3/2}(\epsilon-\mu)\frac{\partial}{\partial\epsilon}n_F(\epsilon) = \frac{1}{3}\frac{gV}{T}\frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int\limits_0^\infty d\epsilon\;n_F(\epsilon)\sqrt{\epsilon}(5\epsilon-3\mu).$$

(c) Die innere Energie

$$U = \sum_\lambda \epsilon_\lambda \langle n_\lambda \rangle = gV \int_0^\infty d\epsilon\ \nu(\epsilon)\ \epsilon\ n_F(\epsilon) = gV \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty d\epsilon\ \epsilon\ n_F(\epsilon)\sqrt{\epsilon}.$$

(d) Die Gesamtteilchenzahl

$$N = \sum_\lambda \langle n_\lambda \rangle = gV \int_0^\infty d\epsilon\ \nu(\epsilon)\ n_F(\epsilon) = gV \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty d\epsilon\ n_F(\epsilon)\sqrt{\epsilon}.$$

(e) Der Ausdruck auf der rechten Seite:

$$U - TS - \mu N = gV \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty d\epsilon\ n_F(\epsilon)\ \sqrt{\epsilon}\ \left\{ \epsilon - \frac{1}{3}(5\epsilon - 3\mu) - \mu \right\}$$

$$= gV \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty d\epsilon\ n_F(\epsilon)\sqrt{\epsilon} \left\{ -\frac{2}{3}\epsilon \right\} = -\frac{2}{3} gV \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty d\epsilon\ n_F(\epsilon)\ \epsilon^{3/2} = \Omega.$$

(f) Die Relation

$$\Omega = -\frac{2}{3} U$$

folgt aus der Ergebnissen (a) und (c).

(g) Von der Thermodynamik:

$$dU = TdS - PdV + \mu dN.$$

Die Legendre-Transformation:

$$d(U - TS - \mu N) = -PdV - SdT - Nd\mu.$$

Deswegen

$$P = -\left(\frac{\partial \Omega}{\partial V} \right)_{T,\mu} = -\frac{2}{3} g \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty d\epsilon\ n_F(\epsilon)\ \epsilon^{3/2} = -\frac{\Omega}{V}.$$

2. Das relativistische entartete Fermi-Gas:

Für Elektronen

$$g = 2.$$

(a)

$$N = \sum_\lambda \langle n_\lambda \rangle = \frac{gV}{2\pi^2\hbar^3} \int_0^{p_F} dp p^2 = V \frac{p_F^3}{3\pi^2\hbar^3} \quad \Rightarrow \quad p_F = \left(3\pi^2\hbar^3\right)^{1/3} \left(\frac{N}{V} \right)^{1/3}.$$

(b)

$$E_F = cp_F \quad \Rightarrow \quad E_F = c\left(3\pi^2\hbar^3\right)^{1/3}\left(\frac{N}{V}\right)^{1/3}.$$

(c)

$$U(T=0) = \sum_\lambda \epsilon_\lambda \langle n_\lambda \rangle = \frac{gV}{2\pi^2\hbar^3}\int\limits_0^{p_F} dp p^2 cp = V\frac{cp_F^4}{4\pi^2\hbar^3}$$

deswegen

$$U(T=0) = \frac{3}{4}c\left(3\pi^2\hbar^3\right)^{1/3} V^{-1/3} N^{4/3}.$$

(d) Laut Definition

$$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}.$$

Bei $T = 0$ gibt es $S = const$. Dann

$$P = -\frac{3}{4}c\left(3\pi^2\hbar^3\right)^{1/3} N^{4/3}\frac{\partial}{\partial V}V^{-1/3} = \frac{1}{4}c\left(3\pi^2\hbar^3\right)^{1/3}\left(\frac{N}{V}\right)^{4/3}.$$

(e) Es folgt aus der Ergebnissen (d) und (c), dass

$$PV = \frac{1}{4}c\left(3\pi^2\hbar^3\right)^{1/3}\left(\frac{N}{V}\right)^{4/3} V = \frac{1}{4}c\left(3\pi^2\hbar^3\right)^{1/3} V^{-1/3}N^{4/3} = \frac{1}{3}U(T=0).$$

(f) Das großkanonische Potential:

$$\Omega = -T\sum_\lambda \ln\left(1+e^{\beta(\mu-\epsilon_\lambda)}\right) = -gTV\int \frac{d^3k}{(2\pi\hbar)^3}\ln\left(1+e^{\beta(\mu-\epsilon_{\mathbf{k}})}\right),$$

$$\Omega = -\frac{TV}{\pi^2\hbar^3}\int\limits_0^\infty dk\, k^2\, \ln\left(1+e^{\beta(\mu-\epsilon_{\mathbf{k}})}\right) = -\frac{TV}{\pi^2c^3\hbar^3}\int\limits_0^\infty d\epsilon\, \epsilon^2\, \ln\left(1+e^{\beta(\mu-\epsilon)}\right),$$

Partielle Integration liefert

$$\Omega = -\frac{1}{3}\frac{V}{\pi^2c^3\hbar^3}\int\limits_0^\infty d\epsilon\frac{\epsilon^3}{1+e^{\beta(\epsilon-\mu)}}.$$

(g) Die innere Energie

$$U = \sum_\lambda \epsilon_\lambda\langle n_\lambda\rangle = \frac{V}{\pi^2\hbar^3}\int\limits_0^\infty dk\, k^2\, \epsilon_{\mathbf{k}} n_F(\epsilon_{\mathbf{k}}) = \frac{V}{\pi^2c^3\hbar^3}\int\limits_0^\infty d\epsilon\frac{\epsilon^3}{1+e^{\beta(\epsilon-\mu)}}.$$

(h) Die Relation

$$\Omega = -\frac{1}{3}U$$

folgt aus der Ergebnissen (f) und (g).