6.3 a

$$H_0 = v_F \begin{pmatrix} 0 & p_x + i p_y \\ p_x - i p_y & 0 \end{pmatrix} ; \quad p_x + i p_y = p\, e^{i\varphi_{\vec p}} = p(\cos\varphi + i \sin\varphi)$$

Eigenvalues: $E_\pm = \pm v_F |\vec p| = \pm v_F \sqrt{p_x^2 + p_y^2}$

Eigenvectors: $\vec v_+ = (e^{i\varphi}, 1)/\sqrt{2}$; $\vec v_- = (-e^{i\varphi}, 1)/\sqrt{2}$.

4x degenerate (spin σ & valley τ).

b

$$H_0 = v_F p \begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix} ; \quad \mathcal{U} = \begin{pmatrix} | & | \\ \vec v_- & \vec v_+ \\ | & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{i\varphi_{\vec p}} & e^{i\varphi_{\vec p}} \\ 1 & 1 \end{pmatrix} .$$

rotation matrix

$$\mathcal{U}^+ H_0 \mathcal{U} = v_F p \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \mathcal{U} = \frac{e^{i\varphi/2}}{\sqrt{2}} \begin{pmatrix} -e^{i\varphi/2} & e^{i\varphi/2} \\ e^{-i\varphi/2} & e^{-i\varphi/2} \end{pmatrix} .$$

$a = \pm 1$

In rotated basis (chiral): $\tilde{H}_0 = U^+ H_0 U = v_F p \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$: $\tilde{H}_0 |p, a\rangle = a v_F p \, |p a\rangle$

$a = \pm 1$: $a = 1$: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $a = 1$: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ eigenfunctions : ~~the~~ Ham. only contains momentum operator $\hat{p}$ $\Rightarrow$ label eigenfunctions by momentum p.

$|\tilde{\lambda}\rangle = \sum_{|\lambda\rangle} |\lambda\rangle \langle \lambda | \tilde{\lambda} \rangle \Rightarrow |\tilde{\lambda}\rangle = |x, AB\rangle$

$|\lambda\rangle = |p, a\rangle$ (chiral basis)

& Define creation operator $\boxed{c^+_{p,a} |0\rangle = |p, a\rangle}$

φ depends on $\vec{p}$.

$\Rightarrow a^+_{\tilde{\lambda}} = \sum_{\lambda} a^+_{\lambda} \langle \lambda | \tilde{\lambda} \rangle$

$\Rightarrow \Psi^+(\vec{x} = R_\mu) = \sum_{p, a = \pm 1} c^+_{p,a} \langle p, a | \vec{x} = R_\mu, A,B \rangle = \frac{1}{\sqrt{2}} \sum_{p,a} e^{-i p R_\mu} \begin{pmatrix} a e^{-i \varphi_p} \\ 1 \end{pmatrix}^T c^+_{p,a}$

2x2 matrix to specify whether e^- is created at A or B site.

momentum does not have any information about sub-unit-cell structure. This is encoded in 2x2 matrix structure.

$|\vec{x}\rangle$ ~~of~~ specifies unit cell of Bravais lattice and basis site A,B via 2x2 matrix structure

$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Mo

6.3 c matrix elements of Coulomb interaction

$$V = \frac{e^2}{2\varepsilon} \sum_{\sigma,\tau} \int d^2x\, d^2y\, \frac{\rho_{\sigma\tau}(\vec{x})\, \rho_{\sigma\tau}(\vec{y})}{|\vec{x} - \vec{y}|} \quad ; \quad \rho_{\sigma\tau}(\vec{x}) = \psi^{\dagger}_{\sigma\tau}(x)\, \psi_{\sigma\tau}(x)$$

$$\varepsilon(\varepsilon_1, \varepsilon_2) = \frac{\varepsilon_1 + \varepsilon_2}{2}.$$

~~Sum over σ,τ yield factor of 4 (spin & valley degeneracy)~~.

$$\Rightarrow V = \frac{\cancel{4}e^2}{2\varepsilon} \int d^2x\, d^2y \sum_{\sigma,\tau} \frac{\psi^{\dagger}_{\sigma\tau}(x)\, \psi_{\sigma\tau}(x)\, \psi^{\dagger}_{\sigma\tau}(y)\, \psi_{\sigma\tau}(y)}{|\vec{x} - \vec{y}|}$$

$$\boxed{\begin{aligned} \psi(x) &= \frac{1}{\sqrt{2}} \sum_{p,a} e^{ipx} \begin{pmatrix} a e^{i\varphi_p} \\ 1 \end{pmatrix} c_{p,a} \\ \psi^{\dagger}(x) &= \frac{1}{\sqrt{2}} \sum_{p,a} e^{-ipx} \left(a e^{-i\varphi_p}, 1 \right) c^{\dagger}_{p,a} \end{aligned}}$$

$$\Rightarrow \rho(x) = \psi^{\dagger}(x)\psi(x) = \frac{1}{2} \sum_{\substack{p,a \\ q,b}} \left(a e^{i\varphi_p}, 1 \right) \begin{pmatrix} b e^{-i\varphi_q} \\ 1 \end{pmatrix} e^{i(p-q)x} \cdot c^{\dagger}_{p,a}\, c_{q,b}$$

$$\Leftrightarrow \boxed{\rho(x) = \sum_{\substack{p,q \\ a,b}} e^{i(p-q)x} \lambda_{(p,a);(q,b)}\, c^{\dagger}_{pa}\, c_{q,b}}$$

$$\boxed{\begin{aligned}\rho(\vec{x}) &= \sum_{\substack{p,a\\ q,b}} e^{i(p-q)x}\,\lambda_{p,a;q,b}\, c^{+}_{pa}\, c_{q,b}\\ \text{with}\quad \lambda_{p,a;q,b} &= \frac{1}{2}\left(1 + ab\, e^{i(\varphi_p - \varphi_q)}\right)\end{aligned}}$$

$$\Rightarrow \quad V = \frac{e^2}{2\varepsilon}\int_{x,y}\int_{\substack{p_1 q_1\\ p_2, q_2}} \sum_{\substack{a_1, b_1\\ a_2, b_2}} \frac{e^{i(p_1-q_1)x}\, e^{i(p_2-q_2)y}}{|\vec{x}-\vec{y}|}\, \underbrace{\lambda_{(p_1 a_1);(q_1,b_1)}}_{\lambda_1}\, \underbrace{\lambda_{(p_2 a_2);(q_2, b_2)}}_{\lambda_2}\, c^{+}_{p_1 a_1} c_{q_1 b_1} c^{+}_{p_2 a_2} c_{q_2 b_2}.$$

Normal order: $\quad c_{q_1, b_1} c^{+}_{p_2, a_2} = - c^{+}_{p_2 a_2} c_{q_1 b_1} + \delta_{q_1 p_2}\delta_{b_1 a_2}$.

Also anti-commute $\quad c_{q_1 b_1} c_{q_2 b_2} = - c_{q_2 b_2} c_{q_1 b_1}$.

Go to relative & absolute coordinates: $\quad \begin{aligned} R &= (x+y)\\ r &= x-y.\end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= \tfrac{1}{2}(R+r)\\ y &= \tfrac{1}{2}(R-r).\end{aligned}$

$$\Leftrightarrow V = \frac{4e^2}{2\varepsilon}\int_{R,r}\int_{\substack{p_1,q_1\\p_2,q_2}}\sum_{\substack{a_1b_1\\a_2b_2}} \frac{e^{i[p_1-q_1+p_2-q_2]\frac{R}{2}}\, e^{i[p_1-p_2-q_1+q_2]\frac{r}{2}}}{|\vec{r}|} \lambda^{a_1b_1}_{p_1q_1}\lambda^{a_2b_2}_{p_2q_2} C^{+}_{p_1a_1} C^{+}_{p_2a_2} C_{q_2b_2} C_{q_1b_1}$$

$$\rightarrow \int_R e^{i(p_1-q_1+p_2-q_2)\frac{R}{2}} = 2\pi\,\delta(p_1-q_1+p_2-q_2)$$

$$\Rightarrow e^{i(p_1-q_1-(q_1-q_2))\frac{r}{2}} = e^{ikr}.$$

<u>variable subst.</u>

$q_1 = q$

$q_2 = p$

$p_1 = q + k$

$p_2 = p - k$

$$\Rightarrow p_1 - p_2 - q_1 + q_2 = q + k - p + k - q + p = 2k$$

$$\Rightarrow V = \frac{e^2}{2\varepsilon}\int_r\int_{p,q,k}\sum_{\substack{a_1,b_1\\a_2,b_2}} \frac{e^{ikr}}{r}\lambda^{a_1b_1}_{q+k,q}\lambda^{a_2b_2}_{p-k,p} C^{+}_{q+k,a_1} C^{+}_{p-k,a_2} C_{p b_2} C_{q,b_1}.$$

<u>Nor.</u> $$\int_r \frac{e^{ikr}}{r} = \int d^2x\, \frac{e^{i\vec{k}\vec{x}}}{|\vec{x}|} = \int_0^\infty dr\, r\int_0^{2\pi} d\theta\, \frac{e^{ikr\cos\theta}}{r} = \int dr\, J_0(kr\cos\theta)$$

$$\int_r \frac{e^{ikr}}{r} = \int_0^\infty dr \int_0^{2\pi} d\theta \; e^{ikr\cos\theta} \underset{\substack{kr = x \\ dr = dx/k}}{=} \int_0^\infty dx \int_0^{2\pi} d\theta \, \frac{1}{k} e^{ix\cos\theta} = \frac{1}{k}\int_0^\infty dx \; 2\pi J_0(|x|) =$$

$$= \frac{2\pi}{k} .$$

$$\Rightarrow \boxed{V = \int_{p,q,k} \sum_{\substack{a_1,b_1 \\ a_2,b_2}} \underbrace{\frac{2\pi \cdot e^2}{2\varepsilon} \frac{1}{|\vec{k}|}}_{= V(\vec{k})} \lambda^{a_1 b_1}_{q+k,q} \lambda^{a_2 b_2}_{p-k,p} \, C^+_{q+k,a_1} C^+_{p-k,a_2} C_{p,b_2} C_{q,b_1}} \qquad V(\vec{k}) = \frac{2\pi e^2}{2\varepsilon |\vec{k}|}$$

(d) Now perform Hartree-Fock approx. at $T=0$ (only lower band is occupied ($a = -1$).

$$C^+_{q+k,a_1} C^+_{p-k,a_2} C_{p,b_1} C_{q,b_2} = \overbrace{\langle C^+_{p-k,a_2} C_{p,b_2} \rangle}^{= \delta(k) \delta_{a_2 b_2} \delta_{b_2,-}} C^+_{q+k,a_1} C_{q,b_1} - \overbrace{\langle C^+_{p-k,a_2} C_{q,b_1} \rangle}^{\delta_{p-k,q} \delta_{a_2 b_1} \delta_{b_1,-}} C^+_{q+k,a_1} C_{p,b_2}$$

$$AB = \langle A \rangle B + A \langle B \rangle - \langle A \rangle \langle B \rangle$$

$$+ \underbrace{\langle C^+_{q+k,a_1} C_{q,b_1} \rangle}_{= \delta(k) \delta_{a_1 b_1} \delta_{a_1,-}} C^+_{p-k,a_2} C_{p,b_2} - \underbrace{\langle C^+_{q+k,a_1} C_{p,b_2} \rangle}_{= \delta_{q+k,p} \delta_{a_1 b_2} \delta_{b_2,-}} C^+_{p-k,a_2} C_{q,b_1}$$

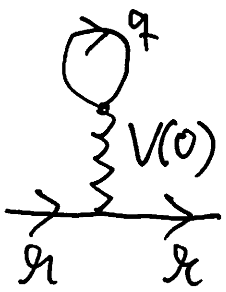
$$- \underbrace{\text{const.}}$$

do not matter for velocity renormalization (only for shift of free energy!

① Hartree-contribution: $\sim \delta_{k,0}$ vanishes due to positive background (Jellium).

e^- move within ~~constant~~ homogeneous pos. background of ions.

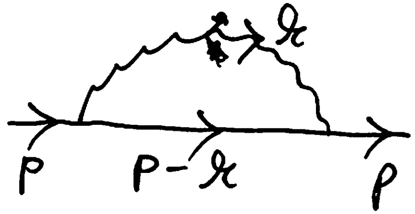
$$\boxed{V(k)\,\delta_{k,0} = 0.}$$



② Fock-contribution:

$$-\langle c^\dagger_{p-k,a_2}\, c_{q,b_1}\rangle\, c^\dagger_{q+k,a_1}\, c_{p,b_1} =$$

$$= -\,\delta_{p-k,q}\,\delta_{a_2 b_1}\,\delta_{b_1 a_1}\; c^\dagger_{q+k,a_1}\, c_{p,b_1}$$



$$\Rightarrow \boxed{V_{\sigma\tau} = -\int_{p,k} \sum_{a_1,b_1} V(k)\left(\lambda^{a_1}_{p,p-k}\,\lambda^{b_2}_{p-k,p} + \lambda^{-b_2}_{p,p-k}\,\lambda^{a_1}_{p-k,p}\right) c^\dagger_{p,a_1}\, c_{p,b_2}.}$$

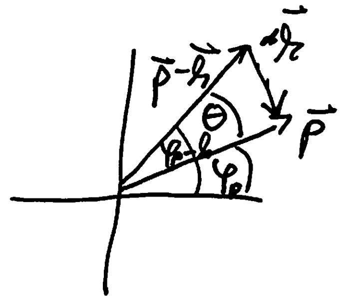
We thus need:

$$\lambda^{a_1}_{p,p-k}\,\lambda^{-b_2}_{p-k,p} = \frac{1}{4}\left(1\mp a_1 e^{i(\varphi_p-\varphi_{p-k})}\right)\left(1\mp b_2 e^{i(\varphi_{p-k}-\varphi_p)}\right) =$$

$$= \frac{1}{4}\Big(\underbrace{1 + a_1 b_2}\ \mp b_2 e^{i(\varphi_{p-k}-\varphi_p)} \mp a_1 e^{i(\varphi_p-\varphi_{p-k})}\Big).$$

const. does not lead to velocity renorm. (need sth. prop. to $|\vec{p}|$.).

Now, $\varphi_{p-k}-\varphi_p$ is the angle between $\frac{\vec{p}-\vec{k}}{|\vec{p}-\vec{k}|}$ and $\frac{\vec{p}}{|\vec{p}|}$.



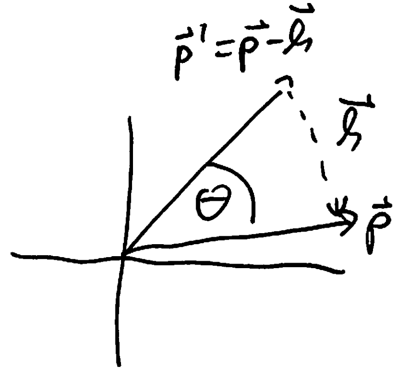
Need: $\int_k V(k)\left(e^{i(\varphi_{p-k}-\varphi_p)} + a_1 b_2 e^{i(\varphi_p-\varphi_{p-k})}\right).$

Write: choose new coordinate system: $\vec{p}=\vec{p}$; $\vec{p}-\vec{k}=\vec{p}\,'$

$$\Rightarrow \int_{\vec{p}\,'} V(\vec{p}-\vec{p}\,')\left(e^{i(\varphi_{\vec{p}\,'}-\varphi_{\vec{p}})} + \underbrace{a_1 b_1}_{=\pm 1.} e^{i(\varphi_{\vec{p}}-\varphi_{\vec{p}\,'})}\right) =$$

Now: (p.6).

$$V(\vec{p}-\vec{p}\,') = \frac{2\pi e^2}{2\varepsilon|\vec{p}-\vec{p}\,'|} = \frac{2\pi e^2}{2\varepsilon\sqrt{p^2+p'^2-2pp'\cos\theta}}$$



$\Rightarrow$ $\boxed{V(\vec{p}-\vec{p}\,') \text{ is even in } \theta}$

$\Rightarrow$ $\boxed{a_1 = b_1}$ so that $(\ldots)$ is also even in θ!

$$\Rightarrow \int_{p'} V(\vec{p}-\vec{p}\,')\, 2\cos\theta = \int \frac{dp'\,p'\,d\theta}{(2\pi)^2} \frac{2\cos\theta \cdot 2\pi e^2}{2\varepsilon\sqrt{p^2+p'^2-2pp'\cos\theta}} =$$

$$= \frac{2e^2}{2\pi\varepsilon\cdot 2}\int_0^{\Lambda} dp'\,p' \int_0^{2\pi} d\theta \frac{\cos\theta}{\sqrt{p^2+p'^2-2pp'\cos\theta}}$$

integrand large when

$$\int d\theta \frac{\cos\theta}{\sqrt{A-B\cos\theta}} = \text{elliptic integrals.}$$

$$\Rightarrow \frac{2e^2}{2\pi\varepsilon\cdot 2}\frac{1}{p}\int_0^{\Lambda} dq\, q\int_0^{2\pi} d\theta\,\frac{\cos\theta}{\sqrt{1+\frac{q^2}{p^2}-2\frac{q}{p}\cos\theta}} \underset{\frac{q}{p}=x}{=} \frac{2e^2}{2\pi\varepsilon\cdot 2}\frac{1}{p}\int_0^{1/p} dx\, p^2\int_0^{2\pi} d\theta\,\underbrace{\frac{x\cos\theta}{\sqrt{1+x^2-2x\cos\theta}}}_{f(x,\theta).}.$$

Look at lower & upper limit to see where dominant contribution of integral comes from:

① $\underline{x \ll 1}$: $\Rightarrow f(x,\theta) \simeq 1-\frac{1}{2}x^2 + x\cos\theta + \ldots \simeq 1 + x\cos\theta + O(x^2)$.

$$\Rightarrow \int_0^{1/p} dx\int_0^{2\pi} d\theta\,\left[x\cos\theta + x^2\cos\theta\right] \quad \underline{\text{converges}}.$$

② $\underline{x \gg 1}$: $\int dx\int d\theta\,\frac{x\cos\theta}{}\left(\frac{1}{x}+\frac{1}{x^2}\cos\theta\right) = \int dx\int d\theta\,\left[\cos\theta + \frac{\cos^2\theta}{x}\right]$.

diverges for $x \to 0$.

$$f(x,\theta) \simeq \frac{1}{x}\frac{1}{\sqrt{\frac{1}{x^2}+1-\frac{2}{x}\cos\theta}} \simeq \frac{1}{x}\cdot\left(1+\frac{1}{x}\cos\theta + O\left(\frac{1}{x^2}\right)\right) = \frac{1}{x}+\frac{1}{x^2}\cos\theta.$$

⇒ approx. integral by divergent contribution:

$$\frac{2e^2}{2\pi\varepsilon\cdot 2}\,p\int_0^{\Lambda/p} dx \int_0^{2\pi} d\theta\ \frac{\cos^2\theta}{x} \approx \frac{2\pi e^2}{2\pi\,\varepsilon\cdot 2}\,p\,\ln\left(\frac{\Lambda}{p}\right)$$

$$\int_0^{2\pi} d\theta\ \cos^2\theta = \pi$$

integral converges at lower bound so it will yield a constant contribution from the lower bound as well!

Altogether (see p. 7):

$$V_{\sigma\tau} = -\int_p\int_q \sum_{a_1,b_1} V(q)\cdot 2\ \lambda^{a_1 \bar{b}_1}_{p,p-q}\,\lambda^{-b_2}_{p-q,p}\ c^\dagger_{p,a_1} c_{p,b_1} =$$

with $\int_q \underbrace{V(q)}_{=\frac{2\pi e^2}{2\varepsilon|q|}} \underbrace{\lambda^{a_1\bar{b}_1}_{p,p-q}\,\lambda^{-b_2}_{p-q,p}}_{=\frac{1}{4}\cdot 2\cos\theta} = \frac{-e^2}{4\cdot 8\,\varepsilon}\,p\,\ln\left(\frac{\Lambda}{p}\right)\delta_{a_1,b_2}$

$= \delta v_F$. (differs by factor 4 from Boris result).

$$\Leftrightarrow V_{\sigma\tau} = +\int_p \sum_a \frac{e^2}{4\cdot 8\,\varepsilon}\,p\,\ln\left(\frac{\Lambda}{p}\right)\cdot 8\ c^\dagger_{p,a} c_{p,a} = +\int_p\sum_a \boxed{\frac{e^2\,\ln\left(\frac{\Lambda}{p}\right)}{4\,\varepsilon\, v_F}}\,v_F\,p\,c^\dagger_{\substack{p,a\\ \sigma,\tau}} c_{\substack{p,a\\ \sigma,\tau}}$$