Karlsruher Institut für Technologie
Institut für Theorie der Kondensierten Materie

Theorie der Kondensierten Materie I WS 2012/2013

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1. Bose-Hubbard model. (15 + 5 + 40 = 60 Punkte)

Bosonic atoms in optical lattices and Cooper pairs in granulated superconductors in certain limits are described by the Bose-Hubbard model,

$$\hat{\mathcal{H}} = U \sum_{\mathbf{r}} (\hat{n}_{\mathbf{r}} - N)^2 - t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \left(\hat{b}^\dagger_{\mathbf{r}} \hat{b}_{\mathbf{r}'} + \hat{b}^\dagger_{\mathbf{r}'} \hat{b}_{\mathbf{r}} \right),$$

where $\hat{b}^\dagger_{\mathbf{r}}$ and $\hat{b}_{\mathbf{r}}$ are bosonic creation and annihilation operators on lattice site $\mathbf{r}$; $\hat{n}_{\mathbf{r}} = \hat{b}^\dagger_{\mathbf{r}} \hat{b}_{\mathbf{r}}$, $U > 0$ is the characteristic interaction strength, $t > 0$– the hopping matrix element, constant N is the average number of particles per site. Only the nearest-neighbour hops are allowed, each bond of the lattice is counted once in the second sum. In this exercise we consider a cubic lattice in a d-dimensional space and a large integer $N \gg 1$. We assume that the particles are spinless.

(a) Find the ground state of the system in absence of the intersite hopping, i.e. at $t = 0$. Find the spectra $E_{\mathbf{k}}$ of the single-boson excitations at small but finite hopping, $U \gg Ntd > 0$.

(b) For $U \gg Ntd \gg T$ evaluate the heat capacitance of the system.

(c) *Superfluid-insulator transition in the mean-field approximation.* Assume, all operators $\hat{b}_{\mathbf{r}}$ and $\hat{b}^\dagger_{\mathbf{r}}$ in the hopping term weakly fluctuate around their average values $\Delta = \langle \hat{b}_{\mathbf{r}} \rangle$ and $\Delta^* = \langle \hat{b}^\dagger_{\mathbf{r}} \rangle$ (mean field approximation). Consider a simplified Hamiltonian which takes into account only the first order fluctuations and neglects the second order. Minimising the corresponding free energy with respect to Δ at $T = 0$ show that the system can be in one of two phases, $\Delta \neq 0$ (superfluid) or $\Delta = 0$ (insulating), depending on the ratio U/t. Calculate the critical ratio $(U/t)_c$ of the superfluid-insulator transition.

2. Fermionic chain. (10 + 30 = 40 Punkte)

Let us consider a system of spinless fermions in a 1D lattice,

$$\hat{\mathcal{H}} = \sum_{i=-\infty}^{+\infty} \left(J_1 \hat{a}^\dagger_i \hat{a}_{i+1} + J_1 \hat{a}^\dagger_{i+1} \hat{a}_i + J_2 \hat{a}_i \hat{a}_{i+1} + J_2 \hat{a}^\dagger_{i+1} \hat{a}^\dagger_i - 2B \hat{a}^\dagger_i \hat{a}_i \right).$$

This Hamiltonian describes, for example, a 1D quantum magnetic system, the so-called XY-model.

(a) Rewrite the Hamiltonian in the momentum representation, using $\hat{a}_n = \int_{-\pi}^{\pi} e^{ikn} \hat{a}_k \frac{dk}{2\pi}$.

(b) (*Bogoliubov-de Gennes transformation.*)
Diagonalise the Hamiltonian and find the quasiparticle spectra E_k.
Hint: it might be useful to consider a transformation of the form

$$\begin{cases} e^{-i\frac{\pi}{4}}\hat{a}_k = u_k\hat{c}_k + v_k\hat{c}^\dagger_{-k} \\ e^{-i\frac{\pi}{4}}\hat{a}_{-k} = -v_k\hat{c}^\dagger_k + u_k\hat{c}_{-k} \end{cases}$$

with real u_k and v_k, satisfying $u_k^2 + v_k^2 = 1$.