Karlsruher Institut für Technologie | Institut für Theorie der Kondensierten Materie

Microscopic Theory of Superconductivity WS 2014/15

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Sheet 01
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1. Thermodynamics of a superconductor (10 + 15 + 15 = 40 points)

(a) Consider a volume V of a system with movable charges but fixed matter. The movable charges experience a force density $\boldsymbol{f} = \rho_f(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c)$. If the charges are moved with velocity $\boldsymbol{v}$, one needs to provide the system with power $-\int d^3r \boldsymbol{f} \cdot \boldsymbol{v} = -\int d^3r \boldsymbol{j}_f \cdot \boldsymbol{E}$.

Derive the change in (internal) energy density du for this case by using the Maxwell equations in matter and by identifying the surface contribution that can be expressed using the Poynting vector $\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{H}$.

The Maxwell equations read $\nabla \cdot \boldsymbol{D} = 4\pi\rho_f$, $\nabla \times \boldsymbol{H} - \frac{1}{c}\dot{\boldsymbol{D}} = \frac{4\pi}{c}\boldsymbol{j}_f$, $\nabla \times \boldsymbol{E} + \frac{1}{c}\dot{\boldsymbol{B}} = 0$ and $\nabla \cdot \boldsymbol{B} = 0$, where the total charge density is $\rho = \rho_f + \rho_P$ with $\rho_P = -\nabla \cdot \boldsymbol{P}$ and the total current density is $\boldsymbol{j} = \boldsymbol{j}_f + \boldsymbol{j}_P + \boldsymbol{j}_M$ with $\boldsymbol{j}_P = \dot{\boldsymbol{P}}$ and $\boldsymbol{j}_M = c\nabla \times \boldsymbol{M}$. Here, $\boldsymbol{P} = \chi_E \boldsymbol{E}$ denotes the polarization with electric susceptibility χ_E and $\boldsymbol{M} = \chi_H \boldsymbol{H}$ the magnetization with magnetic susceptibility χ_H.

(b) Show that using your result from part (a), it follows that the free energy density is given by $df = -sdT + \frac{1}{4\pi}\boldsymbol{H}d\boldsymbol{B}$. Perform a Legendre transformation to the Gibbs free energy density $g(T, H)$ with $H = |\boldsymbol{H}|$. Then determine the difference $g_{n/s}(T, H) - g_{n/s}(T, 0)$ in both the normal state, where $\chi_H \approx 0$, as well as the superconducting state, where you need to consider the Meissner effect.

(c) The critical field strength $H_c(T)$ is defined via the condition that the Gibbs free energy density of the normal and superconducting state are equal for this field value: $g_n(T, H_c) = g_s(T, H_c)$. Express the entropy density difference $s_s(T, H) - s_n(T, H)$ and the specific heat difference $c_{H,s}(T, H) - c_{H,n}(T, H)$ (at constant field H) in terms of this function $H_c(T)$ and determine the order of the phase transition in zero and finite magnetic field.

2. Infinite conductivity and Meissner effect (10 + 10 + 20 + 20 = 60 points)

(a) Consider a homogeneous and isotropic system with $\rho_{ext} = 0$. We are interested in low frequency behavior and thus neglect the contribution $\frac{1}{c}\dot{\boldsymbol{E}}$ in the Maxwell equation $\nabla \times \boldsymbol{B} = \frac{4\pi}{c}(\boldsymbol{j}_{ext} + \boldsymbol{j}_{ind}) + \frac{1}{c}\dot{\boldsymbol{E}}$, where $\boldsymbol{j}_{ind} = \boldsymbol{j}_P + \boldsymbol{j}_M$. Assuming the relation

$$\frac{4\pi}{c}\boldsymbol{j}_{ind}(\boldsymbol{x}, t) = -\int_{-\infty}^{\infty} d^3x' dt' K(\boldsymbol{x} - \boldsymbol{x}', t - t')\boldsymbol{A}(\boldsymbol{x}', t') \tag{1}$$

between the induced current density and the vector potential $\boldsymbol{A}(\boldsymbol{x}, t)$, solve for the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ for given $\boldsymbol{j}_{ext}(\boldsymbol{x}, t)$ in terms of a Fourier integral.

(b) Relate the conductivity $\sigma(\boldsymbol{q}, \omega)$ to the kernel $K(\boldsymbol{q}, \omega)$, where the conductivity relates the induced current density to the electric field via

$$\boldsymbol{j}_{ind}(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} d^3\boldsymbol{x}' dt' \sigma(\boldsymbol{x} - \boldsymbol{x}', t - t') \boldsymbol{E}(\boldsymbol{x}', t') \,. \tag{2}$$

It might be useful to employ the Fourier representation of the Heaviside-θ function

$$\theta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{i}{\omega + i0^+} e^{-i\omega t} \,. \tag{3}$$

(c) Express the real and imaginary part of the conductivity $\sigma(\boldsymbol{q}, \omega) = \sigma'(\boldsymbol{q}, \omega) + i\sigma''(\boldsymbol{q}, \omega)$ in terms of the real and imaginary part of the kernes $K(\boldsymbol{q}, \omega)$. Show that the conductivity $\sigma'(\boldsymbol{q}, \omega)$ becomes infinite if

$$\lim_{\omega \to 0} \lim_{q \to 0} K'(q, \omega) \neq 0 \,, \tag{4}$$

where $q = |\boldsymbol{q}|$.

(d) Neglecting again the contribution from $\frac{1}{c}\dot{\boldsymbol{E}}$, use Maxwell's equations to express the magnetic field $\boldsymbol{H}(\boldsymbol{x}, t)$ as a Fourier integral over the function $\boldsymbol{j}_{ext}(\boldsymbol{q}, \omega)$. Use the result from part (a), to find the magnetic permeability $\boldsymbol{B}(\boldsymbol{q}, \omega) = \mu(\boldsymbol{q}, \omega)\boldsymbol{H}(\boldsymbol{q}, \omega)$. Using that $\mu(\boldsymbol{q}, 0)$ must be positive for zero frequency, show that $q^2 + K'(q, 0)$ may only vanish at $q = 0$ and show that

$$\lim_{q \to 0} \lim_{\omega \to 0} K'(q, \omega) \neq 0 \tag{5}$$

is a sufficient condition for the Meissner effect. Finally, compute the magnetic susceptibility $\chi_H(q, \omega)$ in the case that Eq. (5) is fulfilled.

Infinity conductivity and the Meissner effect thus require different properties of the kernel $K'(q, \omega)$ and do not imply each other.