Karlsruher Institut für Technologie | Institut für Theorie der Kondensierten Materie

Microscopic Theory of Superconductivity WS 2014/15

Prof. Dr. J. Schmalian
Dr. P. P. Orth

Sheet 04
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1. Functional integral for fermions (20 + 20 + 20 + 20 + 20 = 100 points)

If you have a question about the notation please consult Exercise 3.

Consider the partition function for a single fermionic oscillator

$$Z = \mathrm{Tr}\, e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi} = \int \langle -\bar{\psi}|e^{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi}|\psi\rangle e^{-\bar{\psi}\psi} d\bar{\psi}d\psi \tag{1}$$

The path integral for $Z = \mathrm{Tr} e^{-\beta H}$ can be obtained by the following procedure: first write

$$e^{-\beta H} = \lim_{N\to\infty} (e^{-(\beta/N)H})^N = (1-\epsilon H)\cdots(1-\epsilon H) \tag{2}$$

with $\epsilon = \beta/N$. Then introduce the resolution of identity $N-1$ times.

(a) Write down the explicit expression for Z after introducing the identity (see Exercise 3f) $N-1$ times and re-exponentiating, *i.e.*, writing $(1-\epsilon H) \approx e^{-\epsilon H}$.

(b) Define an additional pair of variables (not to be integrated over) for the initial and final Grassmann states (which we denoted $\langle -\bar{\psi}|$ and $|\psi\rangle$ in Eq. (1)): $\bar{\psi}_N = -\bar{\psi}_1$ and $\psi_N = -\psi_1$. Interpret $(\bar{\psi}_{i+1}-\bar{\psi}_i)\psi_i/\epsilon \approx -\bar{\psi}(\tau)\frac{\partial}{\partial\tau}\psi(\tau)$ to arrive at the final expression of the functional integral action S which is defined as $Z = \int \mathcal{D}(\bar{\psi})\mathcal{D}(\psi)e^{-S}$. Give the definition of $\mathcal{D}(\psi)$ as well.

(c) Perform the Fourier expansion $\bar{\psi}(\tau) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\bar{\psi}(\omega_n)e^{i\omega_n\tau}$ as well as $\psi(\tau) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\psi(\omega_n)e^{-i\omega_n\tau}$ with $\omega_n = (2n+1)\pi/\beta$ (why?) to arrive at

$$Z = \int \mathcal{D}\bar{\psi}(\omega)\mathcal{D}\psi(\omega)\exp\Big[\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\bar{\psi}(\omega)(i\omega-\Omega_0+\mu)\psi(\omega)\Big], \tag{3}$$

where we have replaced $\frac{1}{\beta}\sum_n \to \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}$ in the $T\to 0$ limit.

(d) Compute $\langle\bar{\psi}(\omega_1)\psi(\omega_2)\rangle$, $\langle\bar{\psi}(\omega)\psi(\omega)\rangle$ using $2\pi\delta(0) = \beta$ and finally the mean occupation number

$$\langle N\rangle = \frac{1}{\beta Z}\frac{\partial Z}{\partial\mu} = \frac{1}{\beta}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\langle\bar{\psi}(\omega)\psi(\omega)\rangle \tag{4}$$

Does the integral naively converge?

(e) Write down the functional integral for the toy Hubbard model (see Exercise 3 for the definition) and compute the expectation value $\langle N_1\rangle = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\langle\bar{\psi}_1(\omega)\psi_1(\omega)\rangle e^{i\omega 0^+}$ by expanding the expression

$$\langle\bar{\psi}_1(\omega)\psi_1(\omega)\rangle = \frac{\langle\bar{\psi}_1(\omega)\psi_1(\omega)e^{U\int\bar{\psi}_1\bar{\psi}_2\psi_1\psi_2}\rangle_0}{\langle e^{U\int\bar{\psi}_1\bar{\psi}_2\psi_1\psi_2}\rangle_0} \tag{5}$$

to first order in U. Here, $\langle\mathcal{O}\rangle_0$ means averaging with respect to the quadratic part of the action of the toy Hubbard model ($U=0$).