Karlsruher Institut für Technologie — Institut für Theorie der Kondensierten Materie

Microscopic Theory of Superconductivity WS 2014/15

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Sheet 05
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1. Functional integral description of BEC-BCS crossover (10 + 10 + 20 + 10 + 10 + 10 + 10 + 10 + 10 = 100 points)

This question is adressed in the articles of C. A. R. Sa de Melo *et al.*, Phys. Rev. Lett. **71**, 3202 (1993), P. Nozieres and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985), and in M. Y. Veillette *et al.*, Phys. Rev. A **75**, 043614 (2007).

Consider the Hamiltonian of a three-dimensional gas of spinful fermionic particles interacting through an attractive contact pairwise interaction

$$H - \mu N = \sum_{\boldsymbol{k}\sigma} c^\dagger_{\boldsymbol{k}\sigma}(\epsilon_{\boldsymbol{k}} - \mu)c_{\boldsymbol{k}\sigma} - g\int d^3x c^\dagger_\uparrow(\boldsymbol{x})c^\dagger_\downarrow(\boldsymbol{x})c_\downarrow(\boldsymbol{x})c_\uparrow(\boldsymbol{x}) \tag{1}$$

where $\epsilon_{\boldsymbol{k}} = \boldsymbol{k}^2/2m$ and $g > 0$ denotes the strength of the contact interaction.

(a) Write down the imaginary time action $S[\bar{c}, c]$ for this system. It is defined via the partition function $Z = \int \mathcal{D}\bar{c}\mathcal{D}c e^{-S}$.

(b) Perform a Hubbard-Stratonovich decoupling of the interaction term via the identity

$$\begin{aligned}\exp\Big(g\int_0^\beta d\tau\int d^dx c^\dagger_\uparrow c^\dagger_\downarrow c_\downarrow c_\uparrow\Big) \quad (2)\\ = \int\mathcal{D}(\bar{\Delta})\mathcal{D}(\Delta)\exp\Big(-\int_0^\beta d\tau\int d^dx\Big[\frac{1}{g}|\Delta|^2 - \big(\bar{\Delta}c_\downarrow c_\uparrow + \Delta\bar{c}_\uparrow\bar{c}_\downarrow\big)\Big]\Big).\end{aligned}$$

Here, $\Delta(\boldsymbol{x},\tau)$ is a dynamically fluctuation field, and $\bar{c}(\boldsymbol{x},\tau)$ and $c(\boldsymbol{x},\tau)$ denote Grassmann fields. Does $\Delta(\boldsymbol{x},\tau)$ obey periodic or antiperiodic boundary conditions in imaginary time?

(c) Perform a Fourier transformation to momentum space, group the fermionic fields into a so-called Nambu spinor

$$\bar{\Psi}(\boldsymbol{k},\tau) = \begin{pmatrix}\bar{c}_{\boldsymbol{k}\uparrow} & c_{-\boldsymbol{k}\downarrow}\end{pmatrix} \tag{3}$$

$$\Psi(\boldsymbol{k},\tau) = \begin{pmatrix}c_{\boldsymbol{k}\uparrow}\\ \bar{c}_{-\boldsymbol{k}\downarrow}\end{pmatrix} \tag{4}$$

to obtain an action of the form

$$S = \int_0^\beta d\tau\sum_{\boldsymbol{k}}\Big(\frac{1}{g}|\Delta|^2 - \bar{\Psi}G^{-1}\Psi\Big). \tag{5}$$

Determine the inverse Gor'kov Green's function $G^{-1}(\boldsymbol{k},\omega_n)$.

(d) Derive the gap equation

$$\frac{1}{g} = \sum_{\boldsymbol{k}} \frac{\tanh(\xi_{\boldsymbol{k}}/(2T_0))}{2\xi_{\boldsymbol{k}}}, \tag{6}$$

where $\xi_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} - \mu$, by the condition that the non-superconducting $\Delta = 0$ saddle-point of the action becomes unstable, *i.e.*, show that the gap equation follows from the saddle-point equation

$$\frac{\delta S}{\delta \Delta}\bigg|_{\Delta=0} = 0\,. \tag{7}$$

(e) The summation over momentum is performed by switchting to an integration over energy using the density of states. The integral over energies is, however, ultraviolet divergent. It is therefore necessary to introduce a cutoff. In usual BCS theory, this is achieved by the Debye frequency. Here, we wish to study the BEC-BCS crossover where the integral is regularized by the low energy limit of the two-body problem as expressed by the (finite) scattering length. Use

$$-\frac{m}{4\pi a_s} = \sum_{\boldsymbol{k}} \Big[\frac{\tanh(\xi_{\boldsymbol{k}}/(2T_0)}{2\xi_{\boldsymbol{k}}} - \frac{1}{2\epsilon_{\boldsymbol{k}}}\Big] \tag{8}$$

to eliminate the coupling g in the gap equation.

(f) The chemical potential is determined by the number equation $N = -\partial\Omega/\partial\mu$, where $\Omega_0 = S[\Delta = 0]/\beta$. Show that this leads to the number equation

$$n = n_0(\mu, T) = \sum_{\boldsymbol{k}} \Big[1 - \tanh(\xi_{\boldsymbol{k}}/(2T))\Big]\,. \tag{9}$$

The equations (6) and (9) determine the transition temperature T_0 and the chemical potential μ.

(g) In the weak coupling BCS limit, $1/k_F a_s \to -\infty$ (or $g \to 0$) show that the chemical potential is fixed by the density such that $\mu \simeq \epsilon_F$ show that the critical temperature is given by

$$T_0 = \frac{8\gamma}{e^2\pi}\epsilon_F \exp(-\pi(2k_F|a_s))\,, \tag{10}$$

where $\gamma = e^C \approx 1.781$ with Euler constant C.

(h) In the strong coupling limit $1/k_F a_s \to \infty$ ($g \to \infty$) the two equations switch roles. Use the gap equation to fix $\mu = -E_b/2$ with $E_b = 1/ma_s^2$ and the number equation to obtain $T_0 \simeq E_b/[2\ln(E_b/\epsilon_F)^{3/2}]$.

(i) Perform the integration over the fermions in Eq. (5) exactly and expand the resulting expression for small $|\Delta|$ up to fourth order in $|\Delta|$ to obtain a Ginzburg-Landau theory

$$S = S[\Delta = 0] + \sum_{\boldsymbol{q}, i\omega_n} \Gamma^{-1}(\boldsymbol{q}, i\omega_n)|\Delta(\boldsymbol{q}, i\omega_n)|^2 + \int d^d x \frac{u}{4}|\Delta(\boldsymbol{x})|^4\,. \tag{11}$$

with positive constant $u > 0$. Derive the expression for $\Gamma^{-1}(\boldsymbol{q}, i\omega_n)$ and perform the summation over momenta in the strong coupling limit to identify the action as that of weakly interacting bosons of mass $2m$ and density $n/2$. The BEC transition temperature can then be obtained in the approximation of an ideal Bose gas.