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THEORETICAL OPTICS: EXERCISE SHEET 11

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1. Young's double slit interference experiment using a hydrogen lamp - Bohr's model of hydrogen atom

The visible part of the emission spectrum of hydrogen atoms can be used for performing several experiments in optics. In a laboratory, one uses hydrogen lamps, which constitute tubes containing hydrogen gas (Fig. 1). In order to excite the electrons of N hydrogen atoms from the ground state ($n = 1$) to a state of energy $E_n = -13.6/n^2$eV with $n > 1$, a total energy $W = NeV_c$ is provided using an electrical circuit of voltage V_c. The energy per atom $w = W/N = eV_c$ satisfies

$$13.6\left(1-\frac{1}{n^2}\right) \leq w < 13.6\left[1-\frac{1}{(n+1)^2}\right], \tag{1}$$

so that the electron can transit up to the energy level with quantum number n but not $n + 1$. In Eq. 1, the energy w is measured in electron volts (eV).

a. Determine the emission spectrum of a hydrogen lamp, if electric energy of 12.8eV is provided per hydrogen atom. Based on their wave-length, categorize the emitted lines into Lyman, Balmer and Paschen series. (4 points)

b. Calculate the radius of the highest excited state (orbit) that the electron can transit to, for the energy transfer considered in **1a.** (2 points)

c. Determine what is the minimum electric energy per hydrogen atom that we have to provide to the lamp in order to obtain three emission lines in the visible spectrum. (3 points)

d. Assume that for performing Young's double slit interference experiment we make use of a hydrogen lamp. Determine which one of the three emission lines obtained in **1c.** has to be used in order to observe constructive interference of the lowest possible order, for an observation angle $\theta = 30°$ and slit distance $d = 0.262$mm (Fig. 1). (3 points)

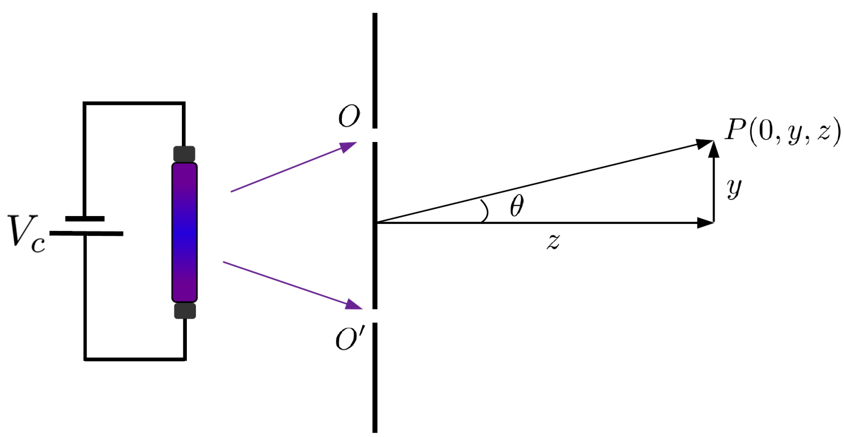

FIG. 1: Young's double slit experiment with two slits located at $\boldsymbol{O}$ and $\boldsymbol{O'}$. The slit distance is d. For the particular setup consider $z >> d, y$. The light source is a hydrogen lamp, which emits light by providing energy using an electrical circuit.

2. Bohr-Sommerfeld quantization and old quantum theory of a harmonic oscillator

Consider a particle of mass m in the presence of a harmonic potential $V(x) = m\omega^2 x^2/2$. The total energy of the particle is

$$E = K + V = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \,, \tag{2}$$

where p corresponds to the momentum of the particle.

Derive the energy spectrum of the quantum harmonic oscillator by applying the Bohr-Sommerfeld quantization condition

$$\oint p dx = nh \,, \tag{3}$$

where the integral is over one period for constant energy E, n is an integer and h corresponds to Planck's constant. (3 points)

3. Quantum theory of a harmonic oscillator

According to modern quantum theory, (x, p, E) become operators $(\hat{x}, \hat{p}, \widehat{\mathcal{H}})$ acting on an object called the state vector $|\Psi\rangle$. If the eigen-states $|x\rangle$ of the position operator $\hat{x}$ are selected as a basis $(\hat{x}\,|x\rangle = x\,|x\rangle)$, then the state vector $|\Psi\rangle$ coincides with $\Psi(x)$, i.e. the time-independent version of the wave-function presented in Sheet **9**, which is a solution of Schrödinger's time-independent equation

$$\widehat{\mathcal{H}}\Psi(x) = \left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right]\Psi(x) \qquad \Rightarrow \qquad \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\Psi(x) = E\Psi(x) \,, \tag{4}$$

since the momentum operator in this basis becomes $\hat{p} = -i\hbar d/dx$. In the case of a harmonic portential $V(x) = m\omega^2 x^2/2$ the energy operator (Hamilton operator or simply Hamiltonian) reads

$$\widehat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \,. \tag{5}$$

a. Verify that the wave-function $\Psi_0(x) = ae^{-bx^2}$ is a solution of Eq. 4 and determine the corresponding energy and constant b. Retrieve constant a by applying the normalization condition

$$\int_{-\infty}^{+\infty} |\Psi_0(x)|^2 dx = 1 \,. \tag{6}$$

The particular wave-function corresponds to the ground state of the system. (4 points)

b. Rewrite the Hamilton operator in terms of the number operator $\widehat{N} = \hat{a}^\dagger \hat{a}$, by introducing the creation $(\hat{a}^\dagger)$ and annihilation $(\hat{a})$ operators, respectively, via the transformation $\hat{x} = \sqrt{\hbar/2m\omega}(\hat{a}^\dagger + \hat{a})$ and $\hat{p} = i\sqrt{m\omega\hbar/2}(\hat{a}^\dagger - \hat{a})$. Retrieve the energy spectrum and comment on the differences compared to the one obtained within the Bohr-Sommerfeld theory in Exercise **2**. (4 points)

c. In the basis of the eigen-states $|n>$ of the number operator $\widehat{N}$, the ground state wave-function $\Psi_0(x)$ corresponds to the state vector $|0>$. Retrieve the wave-function $\Psi_1(x)$, for the first excited state $|1>= \hat{a}^\dagger|0>$. (3 points)

d. Calculate the expectation values (matrix elements)

$$\text{i.} \quad \langle 0|\hat{x}^2|0\rangle \,, \qquad \text{ii.} \quad \langle 0|\hat{p}^2|0\rangle \quad \text{and} \quad \text{iii.} \quad \langle 0|\hat{x}|1\rangle \,, \tag{7}$$

using the creation and annihilation operators and the relations $\hat{a}^\dagger|n>= \sqrt{n+1}|n+1>$ and $\hat{a}|n>= \sqrt{n}|n-1>$. (4 points)