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# THEORETICAL OPTICS: EXERCISE SHEET 2

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## 1. Wave propagation in matter

Maxwell's equations in matter have the form

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho_f\,,\quad \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{B}}=0\,,\quad \boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}}=-\partial_t\boldsymbol{\mathcal{B}}\quad\text{and}\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}_f+\partial_t\boldsymbol{D}\,. \tag{1}$$

**a.** By Fourier transforming $(\boldsymbol{r},t)\to(\boldsymbol{k},\omega)$, rewrite Maxwell's equations in Fourier space (2 points).

**b.** By setting $\rho_f=0$ and $\boldsymbol{J}_f=\boldsymbol{0}$, show that a material with $\boldsymbol{D}(\boldsymbol{k},\omega)=\varepsilon_0\varepsilon(\omega)\boldsymbol{\mathcal{E}}(\boldsymbol{k},\omega)$ and $\boldsymbol{\mathcal{B}}(\boldsymbol{k},\omega)=\mu_0\boldsymbol{H}(\boldsymbol{k},\omega)$ satisfies the constitutive relation (4 points)

$$\omega=\frac{ck}{\sqrt{\varepsilon(\omega)}}\quad\text{with}\quad k=|\boldsymbol{k}|\,. \tag{2}$$

**c.** Retrieve the dispersion relation $\omega=\omega(k)$ for a material with

$$\varepsilon(\omega)=1-\left(\frac{\omega_p}{\omega}\right)^2\,, \tag{3}$$

where $\omega>\omega_p$ and calculate the corresponding phase and group velocities

$$v_p=\frac{\omega}{k}\quad\text{and}\quad v_g=\frac{d\omega}{dk}\,. \tag{4}$$

Observe that $v_p>c$. Does this contradict the constraint that the speed of information transfer should not exceed the speed of light? (6 points)

**d.** By introducing the refractive index $n=c/v_p$, show that (2 points)

$$v_g=\frac{c}{n+\omega\frac{dn}{d\omega}}=\frac{v_p}{1+\frac{\omega}{n}\frac{dn}{d\omega}}\,. \tag{5}$$

## 2. Wave-packets of electromagnetic waves and effects of dispersion

A plane wave has the form $U(x,t)\propto Ae^{i(k_0x-\omega_0t)}$, i.e. constant amplitude and contribution only from a single wave-vector $k=k_0$ ($\omega_0$ is fixed by $k_0$). However, a single plane wave cannot be used for transmitting infromation. Instead, a wave-packet which is a superposition of plane waves must be constructed, that has the form

$$U(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}A(k)e^{i(kx-\omega(k)t)}dk\qquad\text{with}\qquad A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}U(x,t=0)e^{-ikx}dx\,. \tag{6}$$

For most applications, $U(x,t)$ is significantly non-zero in a region of spread $\delta x$ around $x\sim x_0$. This type of profile is called a "pulse". For a spatially wide pulse, only a narrow region of wave-vectors $\delta k\sim 1/\delta x$ around $k_0$ is important.

Consider now the following wave-packet for an electromagnetic wave propagating along the $x$-direction

$$\mathcal{E}_z(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(k) e^{i(kx-\omega(k)t)} dk \,, \tag{7}$$

with the Gaussian profile $A(k) = Ae^{-\frac{(k-k_0)^2}{2\sigma^2}}$ where $k_0$ defines the peak's position, $A$ defines the peak's amplitude and $\sigma$ controls the width. For a spatially wide pulse $\mathcal{E}_z(x, t = 0)$ we have $(k_0/\sigma)^2 >> 1$. In the latter case, the region around $k_0$ affects most significantly the wave propagation and we may approximate the dispersion relation $\omega(k)$ by a truncated Taylor series

$$\omega(k) = \omega_0 + \omega'(k-k_0) + \frac{\omega''(k-k_0)^2}{2} \,, \quad \text{with} \quad \omega_0 \equiv \omega(k_0) \,, \quad \omega_0' \equiv \left.\frac{d\omega(k)}{dk}\right|_{k_0} \quad \text{and} \quad \omega_0'' \equiv \left.\frac{d^2\omega(k)}{dk^2}\right|_{k_0} . \tag{8}$$

Notice that within this approximation, the electric field can be written in the form $\mathcal{E}_z(x,t) = \mathcal{E}_0(x,t)e^{i(k_0x-\omega_0t)}$, with $\mathcal{E}_0(x,t)$ a slowly varying envelope function compared to the fast oscillating exponential term $e^{i(k_0x-\omega_0t)}$.

**a.** Find the expression for the wave-packet $\mathcal{E}_z(x,t) = \mathcal{E}_0(x,t)e^{i(k_0x-\omega_0t)}$ by carrying out the integration over $k$ within the approximation presented in Eq. (8) (5 points). *Hint: Complete the "square", define a shifted integration variable and finally use the Gaussian integral* $\int_{-\infty}^{+\infty} e^{-u^2} du = \sqrt{\pi}$.

**b.** Use Maxwell's equations (with zero charge and current densities) in order to calculate $\boldsymbol{H}(x,t)$ that is produced by the wave-packet $\mathcal{E}_z(x,t) = \mathcal{E}_0(x,t)e^{i(k_0x-\omega_0t)}$ calculated above. Within the slow varying envelope approximation (SVEA), assume that the spatial and time derivatives of the envelope function $\mathcal{E}_0(x,t)$ are negligible compared to the derivatives of $e^{i(k_0x-\omega_0t)}$. Consider linear media with $\boldsymbol{D} = \varepsilon_0\varepsilon\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}} = \mu_0\mu\boldsymbol{H}$ (3 points).

## 3. Linearly and circularly polarized plane waves

**a.** Consider two circularly polarized plane waves with opposite sense of rotation

$$\boldsymbol{\mathcal{E}}_1(\boldsymbol{r},t) = \mathcal{E}_0 \left[\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \varphi_0)\hat{\boldsymbol{x}} + \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \varphi_0)\hat{\boldsymbol{y}}\right] , \tag{9}$$

$$\boldsymbol{\mathcal{E}}_2(\boldsymbol{r},t) = \mathcal{E}_0 \left[\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\hat{\boldsymbol{x}} - \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\hat{\boldsymbol{y}}\right] , \tag{10}$$

with $\varphi_0$ a constant phase. Show that the superposition $\boldsymbol{\mathcal{E}}(\boldsymbol{r},t) = \boldsymbol{\mathcal{E}}_1(\boldsymbol{r},t) + \boldsymbol{\mathcal{E}}_2(\boldsymbol{r},t)$ is a linearly polarized wave (5 points). *Hint: Make use of the relations:*

$$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right) \qquad \text{and} \qquad \sin a - \sin b = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right) . \tag{11}$$

**b.** Show that every elliptically polarized plane wave

$$\boldsymbol{\mathcal{E}}(\boldsymbol{r},t) = \mathcal{E}_x \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\hat{\boldsymbol{x}} + \mathcal{E}_y \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\hat{\boldsymbol{y}} \,, \tag{12}$$

can be written as a superposition of a circularly polarized plane wave and a linearly polarized plane wave (3 points).