Karlsruhe Institute of Technology (KIT) SS2013

Dr. Boris Narozhny, Institut für Theorie der Kondensierten Materie (TKM), narozhny@tkm.uni-karlsruhe.de
Dr. Panagiotis Kotetes, Institut für Theoretische Festkörperphysik (TFP), panagiotis.kotetes@kit.edu

THEORETICAL OPTICS: EXERCISE SHEET 3

Announcement date: 05.05.2013 – Tutorials' date: 09.05.2013 and 10.05.2013

1. Wave propagation in optically anisotropic materials - Fresnel's equation

For optically anisotropic and magnetically non-active lossless media, we have the constitutive relations $\boldsymbol{D}(\boldsymbol{k},\omega) = \varepsilon_0\boldsymbol{\varepsilon}(\omega)\boldsymbol{\mathcal{E}}(\boldsymbol{k},\omega)$ and $\boldsymbol{\mathcal{B}}(\boldsymbol{k},\omega) = \mu_0\boldsymbol{H}(\boldsymbol{k},\omega)$. For these materials, the dielectric constant $\varepsilon(\omega)$ is replaced by the dielectric tensor $\boldsymbol{\varepsilon}(\omega)$. For the particular case we consider

$$\boldsymbol{\varepsilon}(\omega) = \begin{pmatrix} \varepsilon_{xx}(\omega) & 0 & 0 \\ 0 & \varepsilon_{yy}(\omega) & 0 \\ 0 & 0 & \varepsilon_{zz}(\omega) \end{pmatrix} . \tag{1}$$

a. Combine Eq. (1) with

$$\boldsymbol{D}(\boldsymbol{k},\omega) = \varepsilon_0(ck/\omega)^2\left\{\boldsymbol{\mathcal{E}}(\boldsymbol{k},\omega) - [\hat{\boldsymbol{k}}\cdot\boldsymbol{\mathcal{E}}(\boldsymbol{k},\omega)]\hat{\boldsymbol{k}}\right\} , \tag{2}$$

where $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ and $k = |\boldsymbol{k}|$, in order to obtain a homogeneous system of equations for the components of $\boldsymbol{\mathcal{E}}$. For non-zero solutions of $\boldsymbol{\mathcal{E}}$, derive the so called Fresnel's equation, which provides the corresponding dispersion relations $\omega(\boldsymbol{k})$. (3 points)

b. Derive the dispersion relations $\omega(\boldsymbol{k})$ for an optically uni-axial material with $\varepsilon_{xx}(\omega) = \varepsilon_{yy}(\omega) \equiv \varepsilon_\perp$ and $\varepsilon_{zz}(\omega) = \varepsilon_{||}$, where the z-axis defines the optic axis. (3 points)

c. Calculate the group velocity vector $\boldsymbol{v}_g = \frac{d\omega(\boldsymbol{k})}{d\boldsymbol{k}}$, for the dispersion relations found in **1.b**. (2 points)

2. Electromagnetic energy flux density in optically anisotropic materials

Consider the following electromagnetic wave

$$\boldsymbol{\mathcal{E}}(\boldsymbol{r},t) = \mathcal{E}_0\hat{\boldsymbol{n}}\cos\left[\boldsymbol{k}\cdot\boldsymbol{r} - \omega_e(\boldsymbol{k})t\right] , \tag{3}$$

propagating in an optically uni-axial material with $\varepsilon_{xx}(\omega) = \varepsilon_{yy}(\omega) \equiv \varepsilon_\perp$ and $\varepsilon_{zz}(\omega) = \varepsilon_{||}$. The wave propagates with wave-vector $\boldsymbol{k} = \frac{k_0}{\sqrt{2}}(0,1,1)$ according to the extraordinary (e) dispersion relation

$$\omega_e(\boldsymbol{k}) = c\sqrt{\frac{\varepsilon_\perp k_\perp^2 + \varepsilon_{||}k_{||}^2}{\varepsilon_{||}\varepsilon_\perp}} \qquad \text{where} \qquad k_\perp = \sqrt{k_x^2 + k_y^2} \quad \text{and} \quad k_{||} = k_z\,. \tag{4}$$

a. Determine the polarization of the electric field, $\hat{\boldsymbol{n}}$, based on Eq.(2). (4 points)

b. Calculate the average energy flux density

$$\langle\boldsymbol{S}\rangle_T = \langle\boldsymbol{\mathcal{E}}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)\rangle_T = \frac{1}{T}\int_0^T \boldsymbol{\mathcal{E}}(\boldsymbol{r},t)\times\boldsymbol{H}(\boldsymbol{r},t)\,dt\,, \tag{5}$$

over a period $T = 2\pi/\omega_e(\boldsymbol{k})$, for the electromagnetic wave of **2.a**. (4 points)

3. Refraction of electromagnetic waves in optically anisotropic materials

Consider an electromagnetic wave incident to a surface separating vacuum from an optically uni-axial material with $\varepsilon_{xx}(\omega) = \varepsilon_{yy}(\omega) \equiv \varepsilon_\perp$ and $\varepsilon_{zz}(\omega) = \varepsilon_{||}$. For zero surface charge and current densities, the boundary conditions for $\boldsymbol{D}, \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}$ and $\boldsymbol{H}$ read

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = 0\,, \quad \hat{\boldsymbol{n}} \cdot (\boldsymbol{\mathcal{B}}_1 - \boldsymbol{\mathcal{B}}_2) = 0\,, \quad \hat{\boldsymbol{n}} \times (\boldsymbol{\mathcal{E}}_1 - \boldsymbol{\mathcal{E}}_2) = 0 \quad \text{and} \quad \hat{\boldsymbol{n}} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = 0\,, \tag{6}$$

where $\hat{\boldsymbol{n}}$ is the unit vector perpendicular to the surface and $1, 2$ denote the corresponding fields in vacuum and the dielectric respectively.

a. Using the above boundary conditions, retrieve the expressions for the reflected $\boldsymbol{\mathcal{E}}_r$ and transmitted $\boldsymbol{\mathcal{E}}_t$ electric fields. (10 points)

b. Draw the wave-vector surfaces $\omega(\boldsymbol{k}) = \textit{constant}$ for the allowed dispersion relations in the dielectric. Consider both possibilities: **i.** positive uni-axial materials with $\varepsilon_{||} > \varepsilon_\perp$ (for instance quartz SiO_2) and **ii.** negative uni-axial materials with $\varepsilon_{||} < \varepsilon_\perp$ (for instance calcite $CaCO_3$). (4 points)

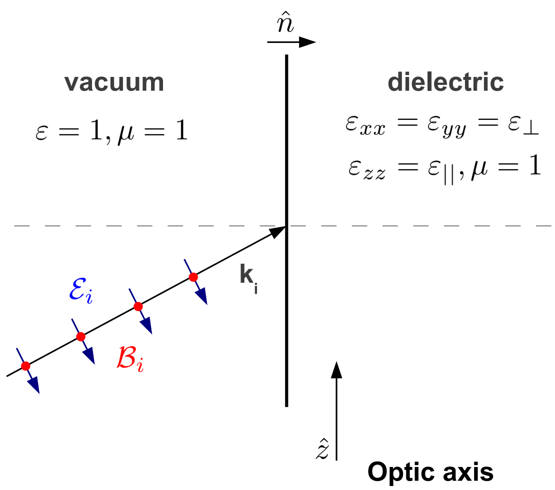

FIG. 1: Refraction of an electromagnetic wave $\boldsymbol{\mathcal{E}}_i$ incident to the surface separating vacuum from an optically active uni-axial dielectric. The polarization of the electric field, $\boldsymbol{k}_i$ and the optic axis lie within the same plane.