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THEORETICAL OPTICS: EXERCISE SHEET 5

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1. Spatial splitting of refracted rays in uni-axial crystals

Consider the refraction of electromagnetic waves on a surface separating vacuum from an optically uni-axial dielectric depicted below:

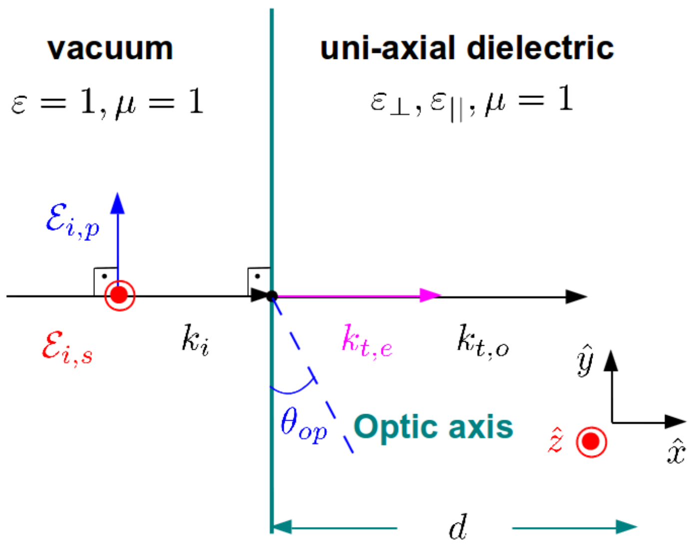

FIG. 1: Refraction of a monochromatic electromagnetic wave $\boldsymbol{\mathcal{E}}_i$ of frequency ω, normally incident to the surface separating vacuum from an optically uni-axial dielectric. The incident beam consists of two orthogonal polarizations $\boldsymbol{\mathcal{E}}_{i,p}||\hat{\boldsymbol{y}}$ and $\boldsymbol{\mathcal{E}}_{i,s}||\hat{\boldsymbol{z}}$. All the wave-vectors, $\boldsymbol{\mathcal{E}}_{i,p}$ and the optic axis lie in the xy-plane. The angle θ_{op} defines the direction of the optic axis.

- **a.** Determine $k_{t,e}$ and $k_{t,o}$. (3 points)
- **b.** Determine the polarization of the electric field corresponding to the ordinary and extraordinary rays. (3 points)
- **c.** Determine the propagation direction of the ordinary and extraordinary rays. (3 points)
- **d.** Calculate the spatial splitting of the ordinary and extraordinary rays along the y-direction, at $x = d$. (3 points)

2. Faraday rotators

Faraday rotators (Fig. 2) constitute a particular class of materials, for which the characteristics of electromagnetic wave propagation through them, can be affected by the simultaneous application of a static and homogeneous magnetic field $\boldsymbol{\mathcal{B}}_s$. In the presence of this constant field, the constitutive relation for the electric induction takes the form $\boldsymbol{D} = \varepsilon_0\varepsilon\left(\boldsymbol{\mathcal{E}} + ig\boldsymbol{\mathcal{B}}_s \times \boldsymbol{\mathcal{E}}\right)$, where g defines the magneto-gyration coefficient. The latter term leads to magneto-optical effects. Specifically, in Faraday rotators clockwise and counter-clockwise circularly polarized electromagnetic waves propagate at different speeds. As a result, the plane of the polarization of an electromagnetic wave which propagates

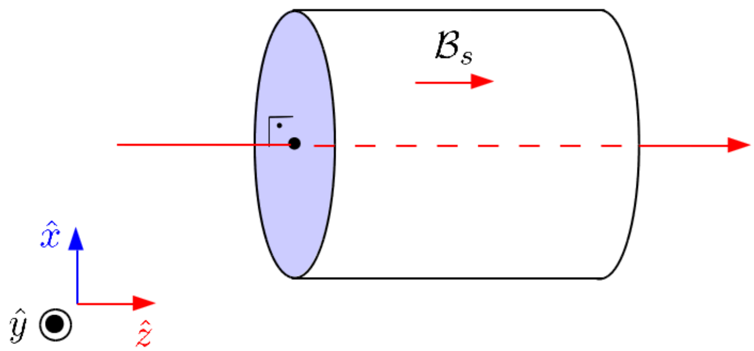

FIG. 2: Faraday rotator: a cylindrically shaped material, for instance TGG (Terbium Gallium Garnet), in an externally constant magnetic field $\boldsymbol{\mathcal{B}}_s$, applied parallel to the cylinder's axis.

through such a material, will rotate. For a typical geometry of a Faraday rotator, the wave-vector of the propagating electromagnetic wave $\boldsymbol{k}$ is parallel to the externally applied constant magnetic field $\boldsymbol{\mathcal{B}}_s$. For the latter case, if we consider $\boldsymbol{k} = k\hat{\boldsymbol{z}}$ and $\boldsymbol{\mathcal{B}}_s = \mathcal{B}_s\hat{\boldsymbol{z}}$, the electromagnetic wave equation reads

$$\left[\left(\frac{ck}{n\omega}\right)^2 - 1\right]\boldsymbol{\mathcal{E}} - ig\mathcal{B}_s\hat{\boldsymbol{z}} \times \boldsymbol{\mathcal{E}} = 0\,, \tag{1}$$

where $\hat{\boldsymbol{z}}$ denotes the unit vector along the z-direction, $n = \sqrt{\varepsilon}$ and the electric field is confined in the plane perpendicular to the wave-vector, satisfying $\boldsymbol{k} \cdot \boldsymbol{\mathcal{E}} = 0$. The above wave-equation provides the following two dispersion relations

$$\omega_\pm(k) = \frac{ck}{n_\pm} \qquad \text{with} \qquad n_\pm = n\sqrt{1 \pm g\mathcal{B}_s}\,. \tag{2}$$

a. By substituting Eq .(2) into Eq. (1), show that the polarization of the electric field which corresponds to the two dispersion relations, is circular. Find the expression for the related magnetic field. (4 points)

b. Consider an electromagnetic wave $\boldsymbol{\mathcal{E}}_i(\boldsymbol{r}, t) = \mathcal{E}_i e^{i(k_i z - \omega t)}\hat{\boldsymbol{x}}$ normally incident to a surface separating vacuum from a Faraday rotator, depicted in Fig. 3. By decomposing the incident beam into two circularly polarized beams, calculate the transmission and reflection coefficents for each circular polarization. Find the expression for the total reflected beam and determine it's polarization. (4 points)

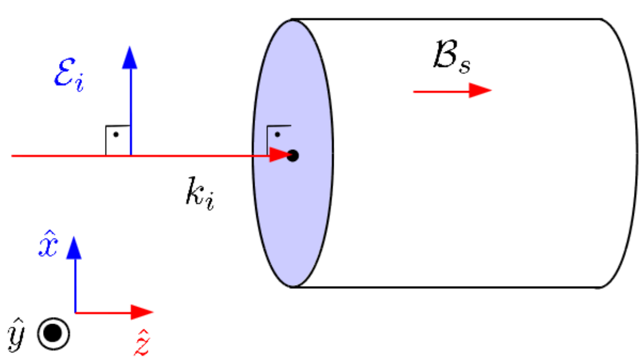

FIG. 3: Refraction of an electromagnetic wave normally incident to the surface separating vacuum from a Faraday rotator. The incident beam is linearly polarized along the x-direction. The wave-vector of the incident beam and the magnetic field applied in the Faraday rotator are both oriented along the z-axis.

3. Optic axis and conical refraction angle in bi-axial crystals

Consider a refracted electromagnetic wave propagating in an optically bi-axial dielectric with propagation wave-vector $k_y = 0$. Assume that $\varepsilon_{xx} < \varepsilon_{yy} < \varepsilon_{zz}$. For the particular wave-vector and frequency ω, the two allowed dispersion relations lead to a circle and an ellipse in the $k_x - k_z$ plane (Fig. 4).

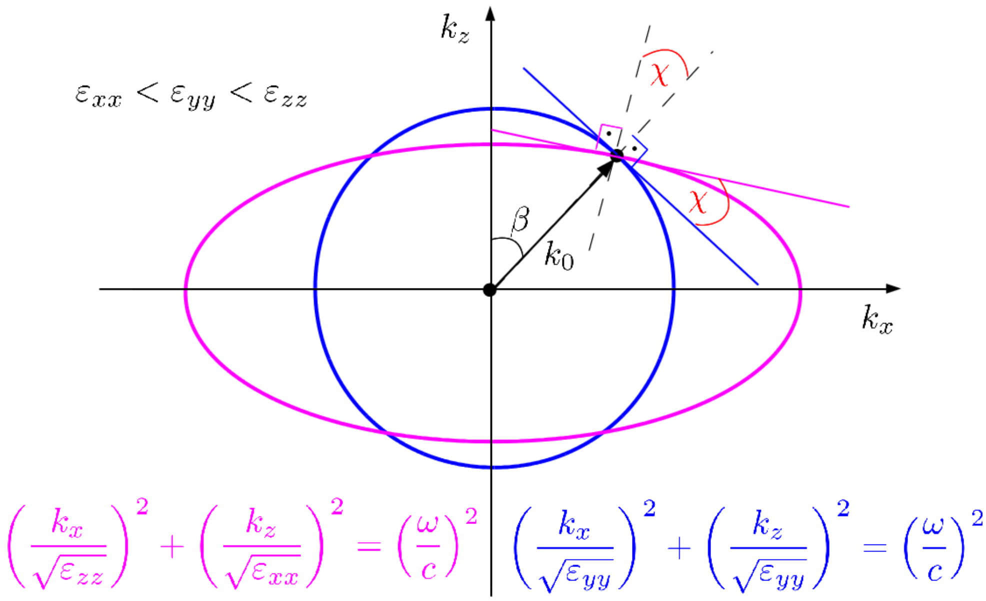

FIG. 4: Wave-vector lines in $k_x - k_z$ plane for a refracted wave in a bi-axial material. The circle and the ellipse corresponding to the two allowed dispersion relations, touch at $\boldsymbol{k}_0$, for which a gradient is not uniquely defined. The latter indefiniteness leads to the phenomenon of conical refraction with angle χ.

a. Determine the angle β and the modulus of the wave-vector $\boldsymbol{k}_0$ for which the circular and elliptical wave-vector lines of Fig. 4 coincide. (4 points)

b. Calculate the angle of conical refraction χ for electromagnetic wave propagation with wave-vector $\boldsymbol{k}_0$. The angle χ is the angle defined by the two tangential lines of the wave-vector lines at $\boldsymbol{k}_0$. (4 points)