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THEORETICAL OPTICS: EXERCISE SHEET 9

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1. Mach-Zehnder interferometer

Consider the Mach-Zehnder interferometer presented Fig. 1 (left). With black we depict perfectly reflecting mirrors. With brown-pink we depict beam-splitters permitting the transmission of a fraction of an incident beam. The brown layer corresponds to the optical coating that is usually made of a dielectric or a metal and the pink part is fabricated by glass, with a typical refractive index $n_g \simeq 1.5$. Here the beam-splitters reflect 50% of any incident beam and form an angle of 45° with the initial beam entering the interferometer (red). Finally, the setup includes two detectors labelled by 1 and 2. Before reaching the detectors the two beams (blue and magenta) merge into a single beam (red).

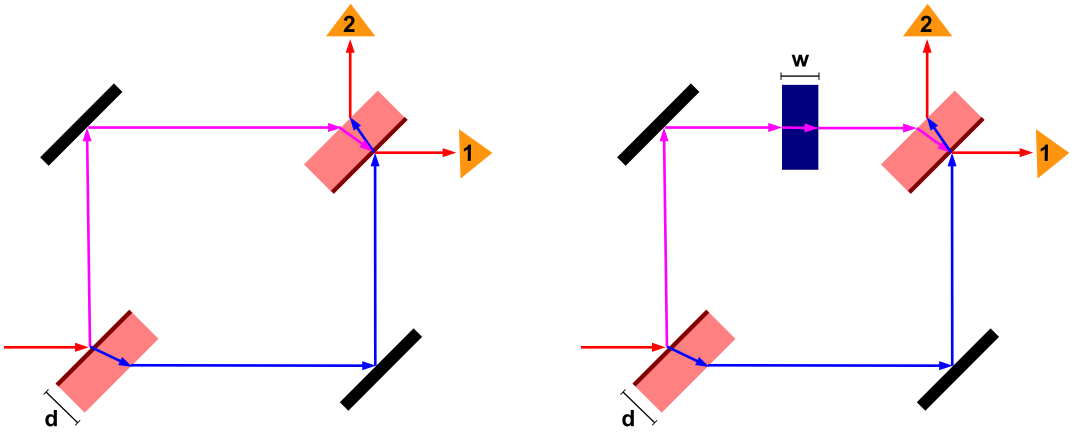

FIG. 1: (Left) Mach-Zehnder interferometer. (Right) Mach-Zehnder interferometric measurement of the refractive index of a sample (blue).

a. Calculate the electric field intensity at the detectors for the setup of Fig. 1 (left). (6 points)

b. Given the Mach-Zehnder interferometric setup, we additionally place a sample with width w and refractive index n_w, as shown in Fig. 1 (right). Calculate the intensity recorded at the two detectors and propose an interferometric method for measuring n_w. (4 points)

2. Interference of two monochromatic waves - Optical beats

Consider two monochromatic plane waves of different frequencies ω_1 and ω_2 which are in-phase at $t = 0$ (Fig. 2).

a. Show that the electric field intensity at the observation point P of Fig. 2 (left) oscillates and determine the so-called "beat" frequency. Determine the time instants for which the intensity demonstrates minima. (5 points)

b. Calculate the electric field intensity at the observation point P' of Fig. 2 (right). Determine the points for which we obtain intensity maxima at a given time instant, if the separation of the two slits is d. (5 points)

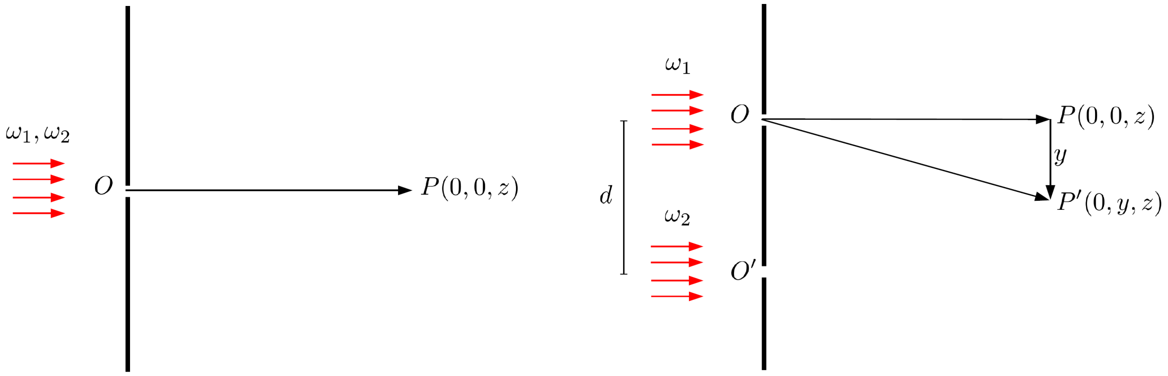

FIG. 2: (Left) Interference of two monochromatic waves (in-phase at $t = 0$) of frequencies ω_1 and ω_2 diffracted on a single slit. (Right) Interference of two monochromatic waves (in-phase at $t = 0$) of frequencies ω_1 and ω_2 diffracted on a double slit. On the upper (lower) slit only waves of frequency ω_1 (ω_2) are incident.

3. Interference of electrons - Aharonov-Bohm effect

In quantum mechanics the wave-particle duality of non-relativistic electrons is captured by Schrödinger's wave equation which in the presence a of time-independent electromagnetic vector potential $\boldsymbol{A}(\boldsymbol{r})$ has the form

$$\frac{\left[\frac{\hbar}{i}\boldsymbol{\nabla}+e\boldsymbol{A}(\boldsymbol{r})\right]^2}{2m}\Psi(\boldsymbol{r},t)=i\hbar\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t}\,, \tag{1}$$

where $e > 0$ is the absolute value of the electron's charge, m is the electron's mass, $\hbar$ corresponds to Planck's constant and $\Psi(\boldsymbol{r},t)$ defines the electron's wave-function. Eq. (1) neglects the intrinsic angular momentum (spin) of the electron.

- **a.** Via the Fourier transform $(\boldsymbol{r},t) \to (\boldsymbol{k},\omega)$, determine the dispersion relation $\omega(\boldsymbol{k})$ in the case $\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{0}$. (2 points)
- **b.** Show that the vector potential can be "removed" from Eq. (1) by performing the gauge transformation $\Psi'(\boldsymbol{r},t) = \mathrm{Exp}\left[-(ie/\hbar)\int_{\boldsymbol{r}_0}^{\boldsymbol{r}}\boldsymbol{A}(\boldsymbol{r}')\cdot d\boldsymbol{r}'\right]\Psi(\boldsymbol{r},t)$, where $\boldsymbol{r}_0$ constitutes an arbitrary point of space. (3 points)
- **c.** Calculate the probability density $|\Psi(\boldsymbol{r},t)|^2$ (analog of intensity) of finding an electron at the observation point P' of Fig. 3, which originates from the superposition of the wave-functions of the two particle beams of frequency ω (energy $\hbar\omega$). Demonstrate that the interference pattern depends on the flux Φ (generated by a magnetic field $\boldsymbol{\mathcal{B}}$) piercing *only* the circular area (radius R) which is not accessible to the two electron beams. The flux dependence of the interference pattern corresponds to the so-called Aharonov-Bohm effect. For the particular setup, consider $(SO) = (SO')$, $(OO') = d$ and assume plane wave incidence on the slits. (5 points)

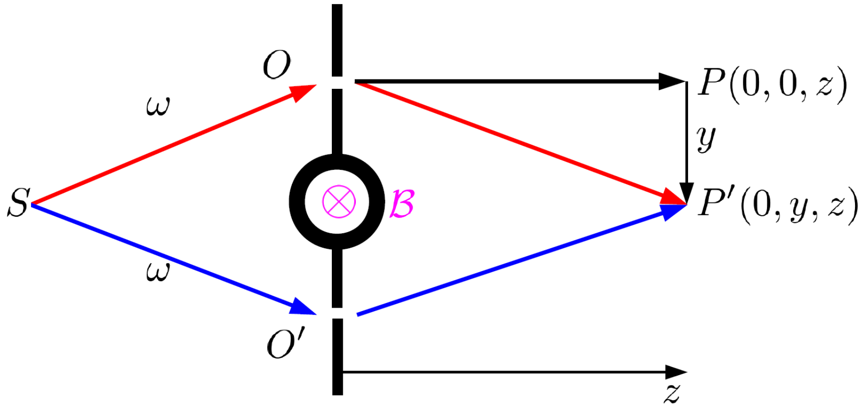

FIG. 3: Aharonov-Bohm setup. Two electron beams interfere at the observation point P' due to : i. the different path distances and ii. the phase shift originating from the finite flux piercing the circular area which is not accessible to the electrons.