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PHOTONS AND PHOTON CORRELATION SPECTROSCOPY

RALPH v. BALTZ
Institut für Theorie der Kondensierten Materie
Universität Karlsruhe, D–76128 Karlsruhe, Germany *

1. Introduction

The majority of optical phenomena and even most of photonics can be well understood on the basis of *Classical Electrodynamics*. The *Maxwell–Theory* is perfectly adequate for understanding diffraction, interference, image formation, photonic–band–gap and negative–index materials, and even most nonlinear phenomena such as frequency doubling, mixing or short pulse physics ... However, spontaneous emission or intensity correlations are not (or incorrectly) captured. For example, photons in a single–mode laser well above the threshold are (counter–intuitively) completely uncorrelated whereas thermal photons have a tendency to "come" in pairs (within the coherence time).
This contribution addresses the following questions:
– Basic properties of photons.
– Quantum description of the Electromagnetic Field (EMF).
– Special photon states.
– Selected optical devices.
– Examples of photon correlations.

To "step into" the field the Paul's easy readable introduction[1], Loudon's classic text[2] and Kidd's [3] historical survey and critical discussion on the evolution of the modern photon will be especially helpful. There are many very good modern textbooks on Quantum Optics available now, e.g. Gerry and Knight[4] or Scully and Zubairy [5]. Bachor[6] discusses basic experiments in Quantum Optics and Haroche and Raimond[7] describe fascinating thought experiments and new concepts of quantum mechanics which now became feasable. In addition there are many proceedings of summer schools and conferences, e.g. Refs.[8–11] which may serve as a resource which almost never runs dry. Perhaps the article of this author in a previous Erice School[12] may be useful, too.

* http://www-tkm.uni-karlsruhe.de

2. Basic Properties of Photons

2.1. PARTICLES AND CORPUSCLES

A *particle* is, by definition, a lossless transport of energy and momentum through free space with a universal *energy–momentum relation*

$$E(\mathbf{p}) = \sqrt{(m_0c^2)^2 + (c\mathbf{p})^2}\,, \quad v(\mathbf{p}) = \frac{\partial E(\mathbf{p})}{\partial \mathbf{p}} = \frac{c^2\mathbf{p}}{E}\,. \tag{1}$$

m_0 is the rest mass of the particle and $\mathbf{v}(\mathbf{p})$ defines the transport velocity of energy and momentum. In addition, a particle may also have charge, spin, angular momentum etc. Examples are electrons, protons or photons. *Bodies (Corpuscles)* are distinguishable, localizable particles with energy–momentum relation (1). In addition they behave as individuums and have a size, shape, elasticity etc. Examples are bullets, golf balls or planets.

2.2. EXPERIMENTAL FACTS

Basic facts of photons are listed in Table I. Based on Lenard's[13] observations on the photo–electric effect and guided by an ingenious thermodynamic approach to describe the black body radiation, Einstein[14] got the vision that the transport of energy of light occurs in form of *light–quanta* "$\hbar\omega$" rather than in a continuous fashion. However, he formulated his idea very reserved:

> *Mit den von Herrn Lenard beobachteten Eigenschaften der lichtelektrischen Wirkung steht unsere Auffassung, soweit ich sehe, nicht im Widerspruch.*
>
> Literal translation:
>
> *Mr. Lenard's observed characteristics of the photoelectric effect is, in our opinion, not inconsistent to our interpretation.*

In a series of intricate experiments, Millikan[15] provided the first experimental verification of the Einstein–hypothesis $E_{\mathrm{kin}} = \hbar\omega - W$ and photoelectric determination of the Planck–constant (as well as the contact potential = difference of work–functions W of cathode and anode).
Further evidence for these "light–quanta", later termed *photons*, arose from the *Compton effect*[16] and the absence of an energy–accumulation time during photo–emission as discovered by Lawrence and Beams[17] and refined later by Forrester et al.[18].
The concept of light–quanta as small corpuscles is in apparent contradiction with typical wave properties like interference fringes and it was expected that such fringes fade out if the intensity of the incident light becomes

TABLE I. Basic properties of photons.

Property	Phenomenon	Quantity	Discoverer (year)
Quantization of energy	photo–electric effect	$E = \hbar\omega$	Einstein (1905) Millikan (1914)
Interference of a photon with itself	persistence of interf. patterns at low intens.		Taylor (1909)
Particle–like transport of energy + momentum	Compton–effect	$E(\mathbf{p}) = c\lvert\mathbf{p}\rvert$	Compton (1923)
Quantum theory of light	unification of wave & particle properties	EMF	Dirac (1927)
Absence of a delay–time for photoelectrons	prompt photoelectrons	$\tau < 3$ns $\tau < 0.1$ns	Lawrence (1928) Forrester (1955)
Photon correlations (thermal light)	Hanbury Brown & Twiss effect	photon coincidences	Hanbury Brown & Twiss (1956)

smaller and smaller so that the probability of having more than a single photon in the spectrometer becomes negligible. Interference experiments at very low intensity were carried out in 1909 by Taylor[19] and later, by Dempster and Batho[20] and Janossy et al.[21]. Yet the interference fringes persisted – a photon interferes (only) with itself, as Dirac[23] said.
In contrast to widespread belief neither the photo–electric effect (cf. Clauser [25]) nor the Compton–effect (cf. Dodd[26]) provide watertight proofs of the photon, yet the wholeness of phenomena is only consistently described within quantum theory. This holds, in particular, for the absence of an energy–accumulation time and the angular dependence of the Compton scattering cross section ("Klein–Nishina–formula"). For a discussion of alternative theories see, e.g., the Rochester Proceedings from 1972[11].

2.3. WHAT IS A PHOTON NOT?

Many contradictory uses exist for the photon[3]. Elementary survey course textbooks usually leave the impression that the photon is a small spherical object which flies on a straight trajectory. A figure like Fig. 1 is dangerous as it pretends that photons in a light beam are tiny corpuscles which have well defined positions. However, as early as in 1909, Lorentz raised the objection that, despite of some striking success of the corpuscular model, one cannot speak of propagating light–quanta concentrated in small regions of space that at the same time remain undivided. He pointed out that the coherence length in interference experiments required a longitudinal extension up to one meter for the photon, while ordinary optical properties demanded a

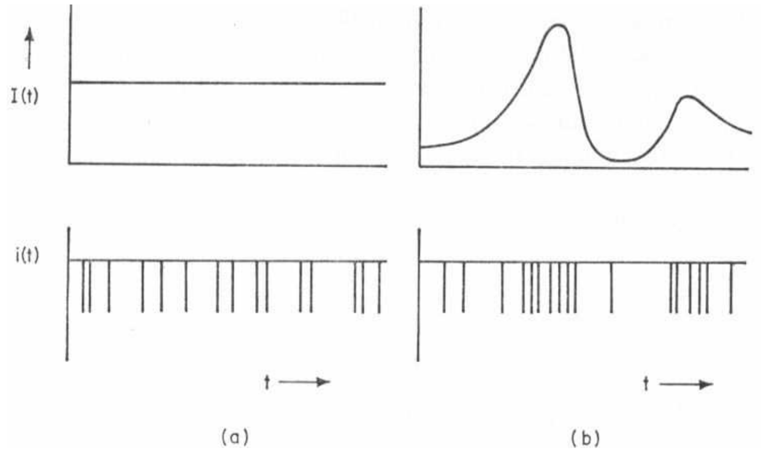

Figure 1. Photoelectric detection. (a) Constant current, (b) fluctuating current. The "combs" refer to the response of a photodetector with high time resolution rather than to "incoming" photons. According to Pike[10].

lateral extension on the order of the diameter of the telescope for light from distant stars. Moreover, a photon cannot be just a scalar "bundle of energy" but has rather definite "vector properties"[22].
In 1921 Einstein complained to Ehrenfest that the problem of quanta was enough to drive him to the madhouse.[1] It was left to Dirac[23] to combine the wave– and particle–like aspects of light so that this description is capable of explaining all interference and particle phenomena of the EMF. We shall follow his traces in Chapt. 4. The answer to the question "What is a photon?" will be left for Sect. 4.1.

3. Basics of Quantum Theory

3.1. CANONICAL QUANTIZATION

Quantum theory provides a very general frame for the description of nature on the microscopic as well as on the macroscopic level. In some cases the extension of a classical theory such as Mechanics or Electrodynamics to a quantum theory can be found along a *correspondence principle.*
First, from the classical theory we have to find (within the Lagrangian formulation, see e.g. Landau and Lifshitz[27] (Vol. I)):

– *Canonical variables* p_i, q_k with *Poisson–brackets* $\{p_i, q_k\} = \delta_{i,k}$, where

$$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial F}{\partial q}, \quad \frac{dG}{dt} = \frac{\partial G}{\partial t} + \{H, G\}.$$

[1] Nobel–prize 1921 for "his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect"!

- Observables (=physical quantities) $G = G(p, q, t)$, in particular the *Hamiltonian* (=energy) $H = H(p, q, t)$.

Then, the corresponding quantum theory is constructed:
- States are described by (normalized) *ket–vectors* $|\psi\rangle$ which are elements of a Hilbert space $\mathcal{H}$ with a scalar product $\langle\psi_1|\psi_2\rangle = (\langle\psi_2|\psi_1\rangle)^*$.
- Canonical (unconstrained) variables $p \to \hat{p}$ $q \to \hat{q}$ and observables $G \to \hat{G}$ are represented by linear, hermitian operators in $\mathcal{H}$

$$\{F, G\} \to \frac{\mathrm{i}}{\hbar}\left[\hat{F}, \hat{G}\right], \quad \text{where} \quad \left[\hat{F}, \hat{G}\right] = \hat{F}\hat{G} - \hat{G}\hat{F}\,, \tag{2}$$

$$\hat{G} \to G(p \to \hat{p}, q \to \hat{q}, t)\,, \quad [\hat{p}_j, \hat{q}_k] = -\mathrm{i}\hbar\delta_{j,k}\,. \tag{3}$$

  Commutators between the p's or the q's themselves vanish.
  (Products of noncommuting operators may be ambiguous).
- The (expectation) value of an observable $\hat{G}$ in state $|\psi\rangle$ is obtained by

$$\langle G\rangle = \langle\psi|\hat{G}|\psi\rangle := \langle\psi|\Big(\hat{G}|\psi\rangle\Big). \tag{4}$$

  $\langle\psi|\hat{G}|\psi\rangle$ can be cast in the form of an *expectation value*

$$\langle\psi|\hat{G}|\psi\rangle = \sum_g gP(g)\,, \quad P(g) = |\langle g|\psi\rangle|^2\,. \tag{5}$$

  $|g\rangle$ denotes an eigenstate of $\hat{G}$ with eigenvalue g: $\hat{G}|g\rangle = g|g\rangle$ and $P(g) > 0$, $\sum_g P(g) = 1$ is the probability to find g in a measurement.
- Dynamics: Initially, the system is supposed to be in state $|\psi_0\rangle = |\psi(t_0)\rangle$. Then, the sequence of states $|\Psi(t)\rangle$ which the system runs through as a function of time is governed by the

$$\text{Schrödinger–Equation} \qquad \boxed{\mathrm{i}\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} = \hat{H}|\Psi(t)\rangle.} \tag{6}$$

  $\hat{H}$ denotes the *Hamiltonian* (=energy) of the system[2].
- Steady states: In a *steady state* values of all observables (which don't explicitely depend on time) are time–independent. Such states exist if $\hat{H}$ is time–independent

$$|\Psi(t)\rangle = e^{-\mathrm{i}Et/\hbar}|\psi\rangle\,, \quad \hat{H}|\psi\rangle = E|\psi\rangle\,. \tag{7}$$

  Different stationary states will be labelled by n, i.e. $E = E_n$, $\psi = \psi_n$.

[2] Eq. (6) holds not only for non–relativistic particles but also for photons.

Pure and mixed states:
Ket–vectors $|\psi\rangle$ describe so–called *pure states* which have zero entropy. They are – loosely speaking – analoga of the mechanical states with fixed q, p or the states of the classical EMF with fixed electrical and magnetic fields ("signals").
Classical statistical states, e.g. particles in thermal equilibrium or thermal radiation ("noise"), are described by a *probability distribution* $P(p, q)$. Such states have nonzero entropy and are called *mixed states*. In quantum theory they are described by a *state–operator* (density operator) $\hat{\rho}$ and (4) is replaced by

$$\langle G\rangle = \text{trace}\ (\hat{\rho}\,\hat{G}). \tag{8}$$

For a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ is a projector onto $|\psi\rangle$.

3.2. HARMONIC OSCILLATOR

As a "warm–up" we consider the (one–dimensional) harmonic oscillator.
Classical Oscillator:

- States are descibed by x, v (or x, p. $v = \dot{x}$, $p = mv$).
- The Newtonian equation of motion reads

$$m\ddot{x} + Dx = F_{ext}(t), \tag{9}$$

  where m, D, F_{ext} denote the mass, spring constant and external force on the particle. The frequency of free oscillations is $\omega_0 = \sqrt{D/m}$.
- Lagrangian: $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}Dx^2 + xF_{ext}$,
- canonical momentum: $p = \partial L/\partial\dot{x} = m\dot{x}$, $\{p, x\} = 1$.
- Hamiltonian: $H = p\dot{x} - L = \dfrac{p^2}{2m} + \dfrac{D}{2}x^2 - xF_{ext}(t)$.

Instead of using real p, x we may also use a complex (dimensionless) amplitude a which turns out to be very useful for quantum mechanics.

$$a = \frac{1}{\sqrt{2}}\Big(\frac{x}{\ell} + \mathrm{i}\frac{p}{m\omega_0\ell}\Big), \quad \{a, a^*\} = \frac{\mathrm{i}}{m\omega_0\ell^2} \to \mathrm{i}/\hbar\,, \tag{10}$$

$$H = \frac{1}{2}m\omega_0\ell^2\Big(a^*a + aa^*\Big) - (a + a^*)f(t)\,, \tag{11}$$

$$\to \hbar\omega_0\frac{1}{2}\Big(a^*a + aa^*\Big) - (a + a^*)f(t)\,, \quad f(t) = \frac{\ell}{\sqrt{2}}F_{ext}(t)\,. \tag{12}$$

With respect to quantum theory the "natural unit of length" $\ell = \sqrt{\hbar/(m\omega_0)}$ has been used. In contrast to (9) $a(t)$ fulfills a first order differential equation

$$\frac{d}{dt}a(t) + \mathrm{i}\omega_0 a(t) = \mathrm{i}f(t)/\hbar\,. \tag{13}$$

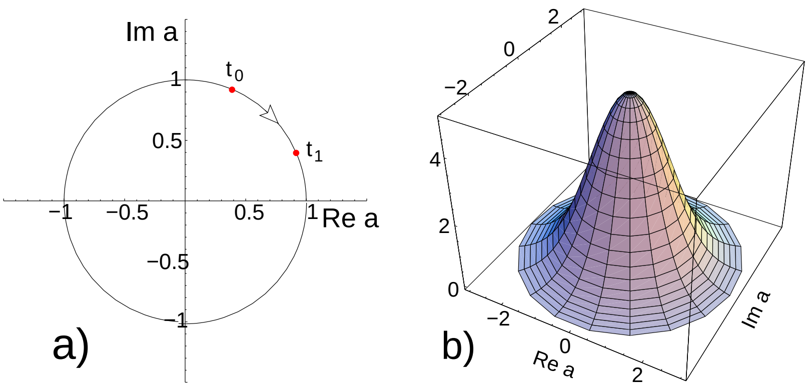

Figure 2. Classical harmonic oscillator. (a) Trajectory in phase space, (b) thermal probability distribution. ($F_{ext} = 0$).

For a free oscillator in thermal contact with a heat bath at temperature T, x and p fluctuate with a *Gaussian* probability distribution, see Fig. 2b,

$$P_{th}(a) = \frac{1}{Z} e^{-H(p,q)/k_B T} = \frac{1}{\pi I_{av}} e^{-|a|^2/I_{av}} \,. \tag{14}$$

Z is the statistical sum (normalization factor), $I_{av} =< |a|^2 >= \hbar\omega_0/k_B T$.
<u>Quantum oscillator:</u>
With canonical operators $p \to \hat{p}$, $x \to \hat{x}$, $[\hat{p}, \hat{x}] = -i\hbar$ and $a \to \hat{a}$, $[\hat{a}, \hat{a}^\dagger] = 1$, we have

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2}{2}\hat{x}^2 - \hat{x}F_{ext}(t) \,, \tag{15}$$

$$\to \hbar\omega_0\Big(\hat{N} + \frac{1}{2}\Big) - (\hat{a} + \hat{a}^\dagger)f(t) \,, \tag{16}$$

$$\hat{N} = \hat{a}^\dagger\hat{a} \,, \quad \hat{N}|n\rangle = n|n\rangle \,, \quad n = 0, 1, 2, \ldots \tag{17}$$

$\hat{N}$ denotes the *number–operator*. The action of $\hat{a}, \hat{a}^\dagger$ on the number–states is

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle. \tag{18}$$

These operators are called *ladder operators* because repeated operation with $\hat{a}, \hat{a}^\dagger$ on $|n\rangle$ creates the "ladder" of all other states: $\hat{a}^\dagger$ "climbs–up", whereas $\hat{a}$ "steps down", see Fig. 3.

$$\hat{a}|0\rangle = 0 \,, \quad |n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\,|0\rangle. \tag{19}$$

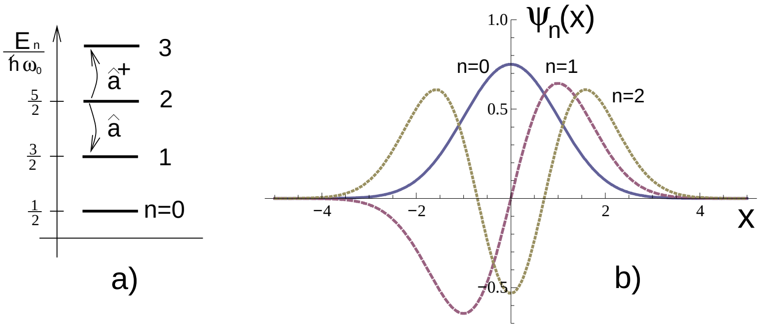

Figure 3. Harmonic oscillator. (a) Energies and (b) stationary wave functions.

Stationary states:
Stationary states are identical with the number–eigenstates $|n\rangle$ and belong to energies $E_n = \hbar\omega_0(n+1/2)$. In position representation $\hat{x} = x$, $\hat{p} = -\mathrm{i}\hbar\frac{\partial}{\partial x}$, $\psi(x) = \langle x|\psi\rangle$ where $|x\rangle$ is an eigenstate of the position operator $\hat{x}$,

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}}\, e^{-x^2/2} H_n(x)\,, \quad E_n = \hbar\omega_0\Big(n + \frac{1}{2}\Big)\,. \tag{20}$$

$H_n(x)$ denote *Hermite–polynomials*, $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2, \ldots$ and $n = 0, 1, 2, \ldots$ (x in units of $\ell = \sqrt{\hbar/(m\omega_0)}$).
Note: The classical oscillator has only a single steady state, $x_0 = 0, p_0 = 0$, whereas a quantum oscillator has infinitely many steady states labelled by $n = 0, 1, 2, \ldots$. In contrast to widespread belief, a single steady quantum state does not correspond to classical motion.
Almost classical states (α–states):
We are looking for states where the expectation values of $\hat{x}$ and $\hat{p}$ vary sinusoidally in time. In addition we require both Δx, Δp to be time–independent and as small as possible. These states were already found by Schrödinger and correspond to a displaced Gaussian ground state wavefunction multiplied by a momentum eigenfunction, see Fig 4. In dimensionless quantities, we have

$$\psi_\alpha(x,t) = \frac{1}{\sqrt[4]{\pi}}\, \exp\left[-\frac{[x - x_c(t)]^2}{2}\right]\, e^{\mathrm{i}xp_c(t)}\, e^{\mathrm{i}\varphi(t)}. \tag{21}$$

$x_c(t)$ and $p_c(t)$ are the solutions of the classical equations of motion of the oscillator (9,13) (even for $f(t) \neq 0$). $\varphi(t)$ is a (irrelevant) time–dependent phase which, however, is needed to solve (6).
In number–representation, these states are given by ($\varphi(t)$ omitted)

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\, \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \quad \alpha = \alpha(t) \to a(t) = [x_c(t) + \mathrm{i}p_c(t)]/\sqrt{2}. \tag{22}$$

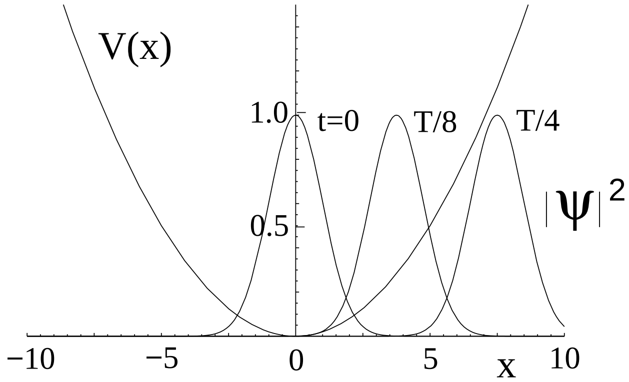

Figure 4. Time development of the coherent state wave function ($\omega_0 T = 2\pi$).

Nowaday, these states are called *coherent states*, *Glauber states*, or just α–*states*. Roy J. Glauber[28] was the first who recognized their fundamental role for the description of optical coherence and laser radiation.[3] Some further details will be discussed in Sect. 5.3.
We shall show in the next section that the EMF is dynamically equivalent to a system of harmonic oscillators, yet for photons wave functions like (20,21) are neither needed nor useful. Instead, only the algebraic properties of $\hat{a}, \hat{a}^\dagger$ will be used.

4. Quantum Theory of Light

4.1. MAXWELL–EQUATIONS

The state of the EMF is described by two (mathematical) vector fields $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}$ which are coupled to the charge and current density of matter $\rho, \mathbf{j}$ by the *Maxwell–Equations*[4]

$$\begin{aligned} \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} - c^2 \mathrm{curl}\, \boldsymbol{\mathcal{B}} &= -\tfrac{1}{\epsilon_0}\mathbf{j}(\mathbf{r},t), \;\text{(a)} \qquad \mathrm{div}\, \boldsymbol{\mathcal{E}} = \tfrac{1}{\epsilon_0}\rho(\mathbf{r},t), \;\text{(c)} \\ \frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} + \mathrm{curl}\, \boldsymbol{\mathcal{E}} &= 0, \qquad\qquad\quad \text{(b)} \qquad \mathrm{div}\, \boldsymbol{\mathcal{B}} = 0. \qquad\quad \text{(d)} \end{aligned} \tag{23}$$

For our purposes a detailed knowledge how to calculate field configurations for specific systems is not required. However, we have to know the relevant dynamical variables of the EMF. Analogous to the interpretation of Classical Mechanics one may view the two differential equations (a,b) with

[3] Nobel prize (1/2) 2005 for “his contribution to the quantum theory of optical coherence”, other half of the prize was given to J. L. Hall and Th. W. Hänsch.
[4] Vectors are set in boldface, electromagnetic fields in calligraphic style.

respect to time as equations of motion of the Maxwell field, whereas (c,d) represent "constraints". Hence, from the 6 components of $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}$ at most $6-2$=4 components are independent dynamical variables at each space point. Therefore, potentials $\Phi, \boldsymbol{\mathcal{A}}$ are more appropriate than $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}$,

$$\boldsymbol{\mathcal{E}} = -\frac{\partial}{\partial t}\boldsymbol{\mathcal{A}}(\mathbf{r},t) - \mathrm{grad}\Phi(\mathbf{r},t), \quad \boldsymbol{\mathcal{B}} = \mathrm{curl}\boldsymbol{\mathcal{A}}(\mathbf{r},t). \tag{24}$$

However, $\boldsymbol{\mathcal{A}}$, Φ are not uniquely determined, rather $\boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{A}}' = \boldsymbol{\mathcal{A}} + \mathrm{grad}\Lambda(\mathbf{r},t), \Phi \to \Phi' = \Phi - \dot{\Lambda}(\mathbf{r},t)$ lead to the same $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}$–fields and, hence, contain the "same physics". $\Lambda(\mathbf{r},t)$ is an arbitrary gauge function. This property is called *gauge invariance* and it is considered as a fundamental principle of nature.
In Quantum Optics (and in solid state physics as well), the *Coulomb gauge*, $\mathrm{div}\boldsymbol{\mathcal{A}} = 0$, is particularily convenient where

$$\begin{aligned} \Delta\Phi(\mathbf{r},t) &= -\frac{1}{\epsilon_0}\rho(\mathbf{r},t), \\ \Delta\boldsymbol{\mathcal{A}}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\boldsymbol{\mathcal{A}}(\mathbf{r},t)}{\partial t^2} &= -\mu_0\mathbf{j}_{tr}(\mathbf{r},t). \end{aligned} \tag{25}$$

$\mathbf{j}_{tr}$ denotes the "transverse" component of the current,

$$\mathbf{j}_{tr}(\mathbf{r},t) = \mathbf{j}(\mathbf{r},t) - \epsilon_0\frac{\partial}{\partial t}\mathrm{grad}\Phi(\mathbf{r},t)\,, \quad \mathrm{div}\mathbf{j}_{tr} = 0\,. \tag{26}$$

Some advantages of the Coulomb gauge are:

– The equations for Φ and $\boldsymbol{\mathcal{A}}$ decouple.
– Φ is not a dynamical system, i.e. Φ is not governed by a differential equation with respect to time, i.e. Φ will not be quantized and there are no "scalar photons".
– As $\mathrm{div}\boldsymbol{\mathcal{A}} = 0$ only 2 of the 3 components of $\boldsymbol{\mathcal{A}}$ are independent variables of the EMF (i.e. there are no "longitudinal photons").

Hence, the EMF has two independent "internal" degrees of freedom at each space point corresponding to two polarization states.

4.2. MODES AND DYNAMICAL VARIABLES

In order to extract the dynamical variables of the EMF from Eq. (25) we decompose the vector potential in terms of *modes* $\mathbf{u}_\ell(\mathbf{r})$

$$\boldsymbol{\mathcal{A}}(\mathbf{r},t) = \sum_\ell A_\ell(t)\,\mathbf{u}_\ell(\mathbf{r}), \tag{27}$$

$$\Delta\mathbf{u}_\ell(\mathbf{r}) + (\frac{\omega_\ell}{c})^2\,\mathbf{u}_\ell(\mathbf{r}) = 0, \quad \mathrm{div}\mathbf{u}_\ell(\mathbf{r}) = 0, \tag{28}$$

$$\int \mathbf{u}^*_\ell(\mathbf{r})\,\mathbf{u}_{\ell'}(\mathbf{r})\,d^3\mathbf{r} = \delta_{\ell,\ell'}. \tag{29}$$

In addition, there will be boundary conditions for $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}$ which fix the eigen-frequencies ω_ℓ of the modes labelled by ℓ. (To lighten the notation we omit the index "tr" from now on). The set of $A_\ell(t)$ can be obtained by using the orthogonality relations (29) of the mode–functions and they represent the generalized coordinates or *dynamical variables* of the EMF which obey the equation of motion

$$\ddot{A}_\ell(t) + \omega_\ell^2 \, A_\ell(t) = \frac{1}{\epsilon_0} j_\ell(t). \tag{30}$$

$j_\ell(t)$ is defined in the same way as $A_\ell(t)$.
From (30) we guess the Lagrangian

$$\begin{aligned} L &= \sum_\ell \frac{1}{2}\,\dot{A}_\ell^2 - \frac{\omega_\ell^2}{2}\,A_\ell^2 + \frac{1}{\epsilon_0}\,j_\ell(t)\,A_\ell\,, \qquad (31) \\ &= \int \left(\frac{\epsilon_0}{2} \boldsymbol{\mathcal{E}}_{tr}^2(\mathbf{r},t) - \frac{1}{2\mu_0} \boldsymbol{\mathcal{B}}^2(\mathbf{r},t) + \mathbf{j}_{tr}(\mathbf{r},t)\ \boldsymbol{\mathcal{A}}(\mathbf{r},t) \right) d^3\mathbf{r}. \end{aligned}$$

The canonical variables are, obviously, $Q_\ell = A_\ell, P_\ell = \dot{A}_\ell$ and the Hamiltonian becomes

$$\begin{aligned} H &= P\dot{Q} - L, \\ &= \int \left(\frac{\epsilon_0}{2} \boldsymbol{\mathcal{E}}_{tr}^2(\mathbf{r},t) + \frac{1}{2\mu_0} \boldsymbol{\mathcal{B}}^2(\mathbf{r},t) - \mathbf{j}_{tr}(\mathbf{r},t)\ \boldsymbol{\mathcal{A}}(\mathbf{r},t) \right) d^3\mathbf{r}. \qquad (32) \end{aligned}$$

In conclusion, each mode of the EMF is equivalent to a driven harmonic oscillator. The state of the EMF is, thus, specified by the set of mode amplitudes A_ℓ and their velocities $\dot{A}_\ell$ at a given instant of time.
$\boldsymbol{\mathcal{A}}(\mathbf{r},t)$ is a real field so that $\mathbf{u}_\ell(\mathbf{r})$ as well as $A_\ell(t)$ ought to be real as well. Nevertheless, the choice of complex modes may be convenient. In particular, in free space we will use "running plane waves" (wave vector $\mathbf{k}$)

$$\mathbf{u}_{\mathbf{k},\sigma}(\mathbf{r}) = \frac{1}{\sqrt{V}}\,\boldsymbol{\epsilon}_{\mathbf{k},\sigma}\,e^{\mathrm{i}\mathbf{k}\mathbf{r}}, \quad \boldsymbol{\epsilon}_{\mathbf{k},\sigma'}^* \cdot \boldsymbol{\epsilon}_{\mathbf{k},\sigma} = \delta_{\sigma,\sigma'}. \tag{33}$$

$\boldsymbol{\epsilon}_{\mathbf{k},\sigma}$ denotes the polarization vector which, by $\mathrm{div}\mathbf{u} = \mathrm{i}\mathbf{k}\cdot\boldsymbol{\epsilon}_{\mathbf{k},\sigma} = 0$, is orthogonal to the wave vector $\mathbf{k}$ (here the notation "transversal" becomes manifest). The two independent polarization vectors will be labelled by $\sigma = 1, 2$. V denotes the normalization volume and, as usual, periodic boundary conditions are implied.
To follow the scheme outlined in the previous section we have to bring the Maxwell–Theory into Hamiltonian form. This is, however, almost trivial because the EMF is dynamically equivalent to a system of uncoupled harmonic oscillators with generalized "coordinates" A_ℓ, see (30).

$$
\begin{aligned}
\boldsymbol{\mathcal{A}}(\mathbf{r},t) &= \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}V}}\Big(a_{\mathbf{k},\sigma}(t)\,\boldsymbol{\epsilon}_{\mathbf{k},\sigma}\,e^{\mathrm{i}\mathbf{kr}}+a^*_{\mathbf{k},\sigma}(t)\,\boldsymbol{\epsilon}^*_{\mathbf{k},\sigma}\,e^{-\mathrm{i}\mathbf{kr}}\Big), \quad (34)\\
\boldsymbol{\mathcal{E}}(\mathbf{r},t) &= -\frac{\partial\boldsymbol{\mathcal{A}}(\mathbf{r},t)}{\partial t}\\
&= \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}V}}\Big(\mathrm{i}\omega_{\mathbf{k}}\,a_{\mathbf{k},\sigma}(t)\,\boldsymbol{\epsilon}_{\mathbf{k},\sigma}\,e^{\mathrm{i}\mathbf{kr}}+cc\Big), \quad (35)\\
\boldsymbol{\mathcal{B}}(\mathbf{r},t) &= \mathrm{curl}\boldsymbol{\mathcal{A}}(\mathbf{r},t)\\
&= \sum_{\mathbf{k},\sigma}\sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}V}}\Big(\mathrm{i}(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k},\sigma})\,a_{\mathbf{k},\sigma}(t)\,e^{\mathrm{i}\mathbf{kr}}+cc\Big), \quad (36)\\
H &= \sum_{\mathbf{k},\sigma}\hbar\omega_{\mathbf{k}}\,a^*_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}-\sqrt{\frac{\hbar}{2\epsilon_0\omega_{\mathbf{k}}}}\Big(j^*_{\mathbf{k},\sigma}(t)\,a_{\mathbf{k},\sigma}+cc\Big), \quad (37)\\
\mathbf{P} &= \int\Big(\frac{1}{\mu_0}\boldsymbol{\mathcal{E}}_{tr}(\mathbf{r},t)\times\boldsymbol{\mathcal{B}}(\mathbf{r},t)\Big)d^3\mathbf{r}=\sum_{\mathbf{k},\sigma}\hbar\mathbf{k}a^*_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}. \quad (38)
\end{aligned}
$$

$\mathbf{P}$ denotes the momentum of the EMF. The complex amplitudes obey the Poisson bracket relations and equation of motion analogous to (10,13)

$$
\begin{aligned}
\{a_{\mathbf{k},\sigma}\,,\,a^*_{\mathbf{k}',\sigma'}\} &= \frac{\mathrm{i}}{\hbar}\,\delta_{\mathbf{k},\mathbf{k}'}\,\delta_{\sigma,\sigma'}\,, \quad (39)\\
\frac{da_{\mathbf{k},\sigma}(t)}{dt}+\mathrm{i}\omega_{\mathbf{k}}\,a_{\mathbf{k},\sigma}(t) &= \mathrm{i}\sqrt{\frac{1}{2\epsilon_0\hbar\omega_{\mathbf{k}}}}\,j_{\mathbf{k},\sigma}(t). \quad (40)
\end{aligned}
$$

Eq. (40) has been already used performing the time–derivative of $\boldsymbol{\mathcal{A}}(\mathbf{r},t)$ in (35).[5]
The complex amplitudes $a_{\mathbf{k},\sigma}$ represent the dynamical variables of the EMF. Its real and imaginary parts are called *quadrature amplitudes* which (apart from numerical factors) are the analogues of position and momentum of a mechanical oscillator, see Sect. 3.2.

[5] The contribution from $\mathbf{j}_{tr}$ drops out in the final result; beware of $a^*_{\mathbf{k},\sigma}\neq a_{-\mathbf{k},\sigma}$ although $j^*_{\mathbf{k},\sigma}=j_{-\mathbf{k},\sigma}$! In contrast to most treatments of the subject no efforts have been made to preserve the "natural sequence" of the amplitudes $a_{\mathbf{k},\sigma}, a^*_{\mathbf{k},\sigma}$. Potential energy/momentum contributions from the scalar potential to Eqs. (37-38) have been omitted. For technical details see Kroll's article in Ref. [8].

4.3. QUANTUM OPTICS

The quantum version of the EMF together with a (nonrelativistic) theory of matter is called *Quantum Optics*. As the EMF is dynamically equivalent to a system of uncoupled harmonic oscillators this task is almost trivial when using ladder operators $a_{\mathbf{k},\sigma} \to \hat{a}_{\mathbf{k},\sigma}$. In the following we shall use the Schrödinger picture, where operators are time–independent.

$$\left[\hat{a}_{\mathbf{k},\sigma}, \hat{a}^{\dagger}_{\mathbf{k}',\sigma'}\right] = \delta_{\mathbf{k},\mathbf{k}'}\ \delta_{\sigma,\sigma'}. \tag{41}$$

The (infinite) zero point energy which arises from the noncommutativity of the $\hat{a}_{\mathbf{k},\sigma}, \hat{a}^{\dagger}_{\mathbf{k},\sigma}$ operators has been omitted in the Hamiltonian as this has no influence on the dynamics of the EMF. The zero point fluctuations of the EMF, however, are still present in the fields, as we shall see later.
The steady states of the infinite set of mode oscillators of the EMF is, thus, labelled by the (infinite set of) quantum numbers $\{n_{\mathbf{k},\sigma}\}$, which individually can take on different nonnegative integers. In free space, the time dependence of $|\{n_{\mathbf{k},\sigma}\}\rangle$ is, as usual, determined by an exponential factor $\exp(-\mathrm{i}n\omega_{\mathbf{k},\sigma}t)$ for each mode. Arbitrary states can be represented as a superposition of these *number–states*, which, therefore, represent a natural basis for the description of the quantum states of the EMF. – But, where are the photons?

4.4. OSCILLATORS AND BOSONS

Dirac[23] has made the important discovery that

> *. . . a system of noninteracting bosons with single particle energies ϵ_ℓ is dynamically equivalent to a system of uncoupled oscillators with frequencies ω_ℓ and vice versa. The two systems are just the same looked at from two different points of view. . .*

Here, *dynamic equivalence* means that all states of an N–boson system which are conventionally described by a symmetric wave function are equally well described in the "oscillator picture", where each single particle state with energy ϵ_ℓ correponds to an oscillator with frequency $\omega_\ell = \epsilon_\ell/\hbar$. Remarkably, the commutation relations $[\hat{a}_\ell, \hat{a}^{\dagger}_{\ell'}] = \delta_{\ell,\ell'}$ between the ladder operators are fully equivalent to the permutation symmetry of the boson wavefunction, and, fortunately, a great deal of notational redundancy in the description of a many–body system is removed. All operators in the "particle picture" (lhs of Fig. 5) can be translated into operators acting on the oscillator states. These operators are conveniently expressed in terms of ladder operators which are now named *creation and destruction operators* for particles because they change the particle number by one. Particle

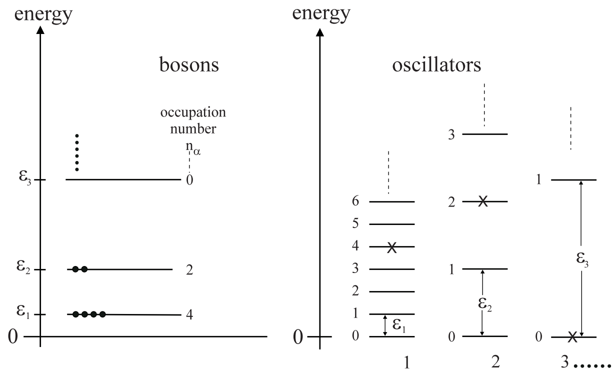

Figure 5. Equivalence of a system of N (noninteracting) bosons with single–particle energies ϵ_ℓ and occupation numbers n_ℓ and an infinite (uncoupled) set of harmonic oscillators with frequencies $\omega_\ell = \epsilon_\ell/\hbar$. Note that the zero–point energies of the oscillators are omitted. Dots symbolize particles, crosses excited states, respectively. ($N = 6$).

conservation implies that all observables contain products of operators with an equal number of $\hat{a}_{\mathbf{k},\sigma}$ and $\hat{a}^\dagger_{\mathbf{k}',\sigma'}$. In contrast to massive particles (in nonrelativistic quantum theory) detection of photons mostly enforces annihilation of them.
The union of all sets of $N = 1, 2 \ldots$ particle subspaces plus the $N = 0$ “no particle” state (vacuum)

$$|0\rangle = |0, 0, 0, \ldots\rangle$$

is called *Fock space*. The number states $|\{n_\ell\}\rangle$ are the eigenstates of the *particle number operator*

$$\hat{N} = \sum_\ell \hat{a}^\dagger_\ell \, \hat{a}_\ell. \tag{42}$$

Now, the particle number itself becomes a dynamical variable and we can even describe states which are not particle number eigenstates of the system. The Fock representation is also called *occupation number representation* or “second quantization”. It is much more flexible than the original formulation with a fixed particle number.

4.5. WHAT IS A PHOTON?

The bosons corresponding to the quantized oscillators of the EMF are called *photons*.

- Photons are the eigenstates of the number operator $\hat{N}$. In particular, there are photon number–eigenstates belonging to each mode, $|\{n_{\mathbf{k},\sigma}\}\rangle$ with integer $n_{\mathbf{k},\sigma} = 0, 1, 2 \ldots$

- A single photon belonging to a fixed mode $(\mathbf{k}, \sigma)$ is also eigenstate of the Hamiltonian with energy $E = \hbar\omega_{\mathbf{k},\sigma}$ and momentum $\mathbf{p} = \hbar\mathbf{k}$ with $E(\mathbf{p}) = c|\mathbf{p}|$.
- Photons have two polarization degrees of freedom, e.g. there are two orthogonal linear polarizations $\sigma = 1, 2$ or left/right circular polarization base states
$$|1_\pm\rangle = \frac{1}{\sqrt{2}}\Big(|1_{\mathbf{k},1}\rangle \pm \mathrm{i}|1_{\mathbf{k},2}\rangle\Big).$$
According to the prevailing belief, the spin of the photon[6] or some other particle is a mysterious internal angular momentum for which no concrete physical picture is available, and for which there is no classical analog. However, it can be shown that the spin of the photon may be regarded as an angular momentum generated by a circulating flow of energy in the wave field, see the nice article by Ohanian[29].
- All types of classical interference phenomena are automatically captured by the mode structure of (34).
- Localization of energy arises as the outcome of a measurement which causes the state of the EMF to "collapse" into an eigenstate of the measuring device as a result of a position measurement, e.g. the absorption of a photon by an atom or in a pixel of a CCD–camera.

Today our interpretation of photons differs substantially from the original idea of small energy "bullets" or "darts"[22]. Although photons (in free space) have a definite energy–momentum relation, photons are not "objects" in the sense of individual, localizable classical corpuscles. By contrast, they are nonlocalizable, indistinguishable, obey Bose–statistics, and they can be created and annihilated easily. Moreover, in most cases the relevant photon states are not states with a fixed photon number. This will become obvious when discussing various examples in the next sections.

5. Special Photon States

In the following we shall discuss some special states of the EMF and their expectation values of the $\mathcal{E}$, $\mathcal{B}$, energy, and momentum. The physically relevant states cannot be eigenstates of the electrical field operator $\hat{\mathcal{E}}$ as these have infinite energy. ($\hat{\boldsymbol{\mathcal{E}}}$ corresponds to the position or momentum of a mechanical oscillator).
The "quantum unit" of the electrical field strength is $\mathcal{E}_0 = \sqrt{\hbar\omega/2\epsilon_0 V}$. For green light, $\lambda = 500$nm, and a quantization volume of $V = 1\mathrm{cm}^3$,

6 Although photons are spin–1 particles there are only but two spin projections (parallel and antiparallel to momentum), i.e. photons are characterized by helicity. For massless particles there is no rest frame.

$\mathcal{E}_0 \approx 0.075\text{V/m}$, whereas, in a microresonator of linear dimension $1\mu\text{m}$, $\mathcal{E}_0 \approx 7.5 \times 10^4\text{V/m}$!

5.1. N-PHOTONS IN A SINGLE MODE

We consider n photons in a single mode with $\mathbf{k} = (k, 0, 0)$ and linear polarization along y–direction $\boldsymbol{\epsilon}_{\mathbf{k},\sigma} = (0, 1, 0)$. $|n\rangle$ is, of course, an eigenstate of the photon number operator (42). (Mode indices $\mathbf{k}, \sigma$ are omitted, for brevity).
Without a driving current source this state is an eigenstate of the Hamiltonian with energy $n\hbar\omega$ and it evolves in time according to

$$|n; t\rangle = |n\rangle \, e^{-\mathrm{i}n\omega t}. \tag{43}$$

In addition, this state is also a momentum eigenstate with eigenvalue $n\hbar\mathbf{k}$, see (38). However, $|n\rangle$ is not an eigenstate of the electrical field operator, Eq. (35), because $\hat{a}_{\mathbf{k},\sigma}, \hat{a}^\dagger_{\mathbf{k},\sigma}$ changes the number of photons by ± 1. In particular, we have

$$\langle n; t|\hat{\mathcal{E}}(\mathbf{r})|n; t\rangle \;=\; 0, \tag{44}$$

$$\langle n; t|\hat{\mathcal{E}}^2(\mathbf{r})|n; t\rangle \;=\; \mathcal{E}_0^2 \, (2n + 1). \tag{45}$$

Certainly, such a state does not correspond to a classical sinusoidal wave, instead it is pure “quantum noise”. Note, even in the vacuum state $|0\rangle$, *zero point fluctuations* of $\mathcal{E}, \mathcal{B}$ are present.

5.2. SINGLE PHOTON WAVE PACKET

We consider a superposition of one–photon states with are composed of different modes (but with the same polarization).

$$|\phi_\sigma(\mathbf{k}); t\rangle = \sum_{\mathbf{k}} \phi_\sigma(\mathbf{k}) \, e^{-\mathrm{i}\omega_{\mathbf{k}} t} \, |1_{\mathbf{k},\sigma}\rangle. \tag{46}$$

$\phi_\sigma(\mathbf{k})$ is an arbitrary, normalized function which, with some care, may be interpreted as a wave function of a photon (–wave packet) in momentum space. However, there is no well defined photon position representation, see, e.g. Landau–Lifshitz[27] (Vol. 4a). The question of localization of photons is discussed, e.g., by Clauser in Ref.[11].
Grangier et al.[30] reported an interesting phenomenon where two Ca atoms shared a single photon similar to a two–slit diffraction experiment. Photodissociation of a diatomic homonuclear molecule A_2 (Ca_2) yielding two atoms recoiling in opposite directions, one in excited state (A^*) and the other in the ground state (A). Either atom can actually be excited, and

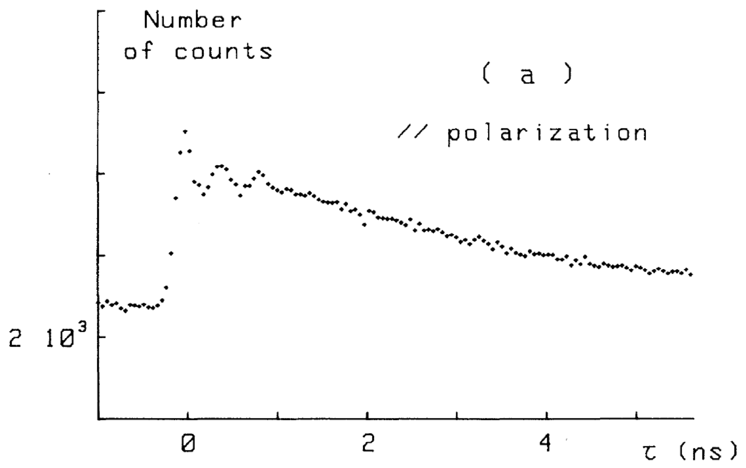

Figure 6. Two Ca–atoms radiate a single photon. Number of detected fluorescent photons (same polarization as the laser light) as function of time delay τ (50ps per channel). According to Grangier et al.[30].

subsequently reemit a photon at the atomic frequency ω_0 , so one must consider two undishinguishable paths for the whole process

$$\hbar\omega_L + A_2 \rightarrow \left\{ \begin{matrix} A & + A^* \\ A^* + A & \end{matrix} \right\} \rightarrow 2A + \hbar\omega_0 \, .$$

$\hbar\omega_L$ refers to the photodissociating light. Interference oscillations in Fig. 6 originate from two recoiling atomic dipoles at distance $d = 2v\tau$, where $v \approx$ 1100m/s is the velocity and τ time of observation. By using a tunable laser, one could thus obtain interesting information about the "energy–landscape" of the dissociating molecular state.

5.3. COHERENT STATES (IDEAL SINGLE MODE LASER)

We are looking for a state, in which the $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}$ fields vary sinusoidally in space and time with $\boldsymbol{\mathcal{E}}$–uncertaincy as small as possible, i.e., have time independent uncertaincies in the quadrature amplitudes with $\Delta X_1 = \Delta X_2 = 1/2$ and $\Delta X_1 \Delta X_2 = 1/4$. We have already discussed these states in Sect. 3.2

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \, |n\rangle. \tag{47}$$

α–states, also termed *Glauber states* or *coherent states* have a number of interesting properties[5]:

– $|\alpha\rangle$ is an eigenstate of the destruction operator

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{48}$$

where α is an arbitrary complex number.

– α–states can be generated by the unitary "displacement" operator $\hat{D}$

$$
\begin{aligned}
|\alpha\rangle &= \hat{D}(\alpha)|0\rangle, & (49)\\
\hat{D}(\alpha) &= e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-\frac{1}{2}|\alpha^2|} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}, & (50)\\
\hat{D}^\dagger\, \hat{a}\, \hat{D} &= \hat{a} + \alpha. & (51)
\end{aligned}
$$

– The time dependence of an α–state (even with a classical external current source) is obtained just by replacing $\alpha \to \alpha(t)$,

$$|\alpha, t\rangle = e^{\mathrm{i}\phi(t)}\, |\alpha(t)\rangle, \qquad (52)$$

where $\alpha(t) \to a(t) = [x_c(t) + \mathrm{i}p_c(t)]/\sqrt{2}$ correspond to the position and momentum of a classical oscillator. For a free oscillator $\alpha(t) = \alpha\, e^{-\mathrm{i}\omega t}$.

– Although the α–eigenvalues form a continuous spectrum, they are normalizable and (over—) complete but not orthogonal. These states form a convenient basis for the description of "almost classical states" of the EMF for which the state–operator can be solely represented by its diagonal elements

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha|\, d^2\alpha. \qquad (53)$$

For example, for thermal states the Glauber P–function is $P(\alpha) = \frac{1}{\pi\bar{n}} \exp(-|\alpha|^2/\bar{n})$, where $\bar{n}$ is the mean photon number in the mode, see also (14) and Fig. 2b.

Expectation values and uncertainties of the electrical field and photon number and the probability to measure n photons are [$\alpha = |\alpha| \exp(\mathrm{i}\phi)$, polarization index omitted]:

$$
\begin{aligned}
\mathcal{E}(\mathbf{r}, t) &= \langle\alpha(t)|\hat{\mathcal{E}}(\mathbf{r})|\alpha(t)\rangle = -2\mathcal{E}_0|\alpha|\, \sin(\mathbf{kr} - \omega_{\mathbf{k}}t + \phi), & (54)\\
(\Delta\mathcal{E})^2 &= \mathcal{E}_0^2, & (55)\\
\langle\hat{N}\rangle &= |\, \alpha\, |^2 = \bar{n}, \quad (\Delta\hat{N})^2 = \langle\hat{N}\rangle, & (56)\\
p_n &= |\langle n|\alpha\rangle|^2 = e^{-\bar{n}}\, \frac{\bar{n}^n}{n!}. & (57)
\end{aligned}
$$

– α is not an eigenstate of the photon number operator.

– The photon distribution function p_n is a *Poissonian* with mean photon number $\bar{n} = |\alpha|^2$ and uncertainty $(\Delta N)^2 = \bar{n}$.

– The relative amount of fluctuations in the electrical field decreases with increasing amplitude, $\Delta N/\langle N\rangle = 1/\sqrt{\langle\bar{n}\rangle}$.

– Thus, in a coherent state photons behave like uncorrelated classical objects! In contrast to naive expectations, the photons in a (single mode)

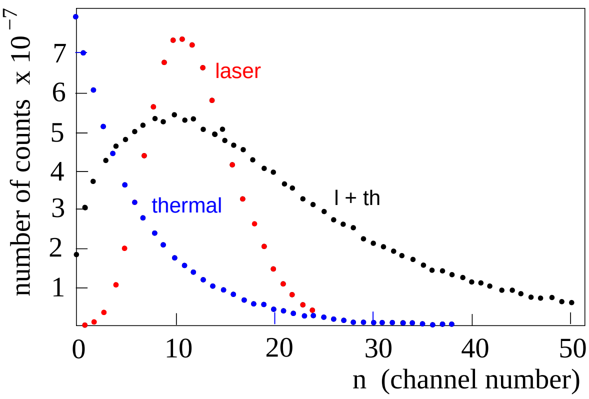

Figure 7. Photon count distribution for a single mode laser, thermal light, and mixing of both. According to Arecchi, in Ref.[9].

laser (and well above the threshold) "arrive" in a random fashion, in particular they do not "ride" on the electrical field maxima. See also Fig. 1.

How to generate α–states? As α–states are eigenstates of the (nonhermitian) destruction operator $\hat{a}$, there is no observable with a corresponding "apparatus" to create these states just by doing a measurement! However, α–states can be simply generated from the vacuum by a classical current source (e.g. antenna of a radio transmitter)

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} - \left[f(t)\hat{a}^\dagger + f^*(t)\hat{a} \right], \tag{58}$$

where $f(t) \propto j(t)$. Nevertheless, it was a great surprise that the light–matter interaction in a laser (well above the threshold) could be modelled in such a simple way.
The amplitude of the electrical field of a laser may be well stabilized by saturation effects, but there is no possibility to control the phase, i.e., a more realistic laser state would be described by the density operator

$$\hat{\rho} = \int \frac{d\phi}{2\pi} \mid \alpha\rangle\langle\alpha \mid = \sum_n p_n \, |n\rangle\langle n|. \tag{59}$$

This state is made up of a mixture (incoherent superposition) of n–photon states with a *Poissonian distribution.* [A simple model to describe laser light with a finite linewidth which is caused by phase diffusion has been given by Jacobs[31].]

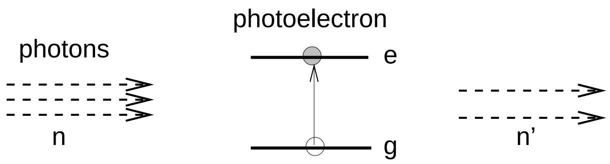

Figure 8. Principle of a photodetector. Absorption of a photon induces a transition from a bound electron state to an excited "free" state which leads to a current (–impulse).

5.4. THERMAL (CHAOTIC) PHOTON STATES

A single mode of thermal (black body) radiation is described by the statistical operator

$$\hat{\rho} \;=\; \frac{1}{Z}\, e^{-\beta \hat{H}} = \sum_{n=0}^{\infty} b_n \, |n\rangle\langle n|, \tag{60}$$

$$b_n \;=\; \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} = \frac{1}{1+\bar{n}} \left(1 + \frac{1}{\bar{n}}\right)^{-n} = \frac{1}{Z}\, e^{-\beta\hbar\omega n}\,, \tag{61}$$

$$\langle \hat{N} \rangle \;=\; \mathrm{tr}(\hat{\rho}\, \hat{N}) = \frac{1}{e^{\beta\hbar\omega} - 1} = \bar{n}, \quad (\Delta\hat{N})^2 = \bar{n}(\bar{n}+1). \tag{62}$$

$\beta = 1/(k_B T)$, $Z = 1/(1 - \exp(-\beta\hbar\omega))$ is the partition function, b_n is the *Bose–Einstein* photon distribution (geometric sequence), and $\bar{n}$ is the mean photon number in the mode, see Fig. 7. In contrast to coherent states, $\Delta\hat{N}/\bar{n} \to 1$ for $\bar{n} \to \infty$.

Problem P1: A provoking question:
"Are there photons in static electric and magnetic fields?"
(The answer will be given at the end of the article.)

6. Detectors and Optical Devices

Passive optical components like phase shifters, lenses, mirrors, polarizers etc., are used in Quantum Optics to change modes (base transformation) whereas a photodetector gives a "click" by absorbing a photon.

6.1. PHOTODETECTORS

In a classical description the electrical current of a photodetector is proportional to the light intensity (energy density) averaged over a cycle of oscillation[2]

$$J_{\mathrm{PD}} \propto \langle \mathcal{E}(t)^2 \rangle_{\mathrm{cycl}} \;\propto\; \mathcal{E}^{(-)}(t)\, \mathcal{E}^{(+)}(t). \tag{63}$$

$\mathcal{E}^{(\pm)}$ denote the positive(negative) frequency parts of the electric field, e.g. $\boldsymbol{\mathcal{E}}^{(+)} \propto \exp(-\mathrm{i}\omega t)$ (polarization properties and space variables have been

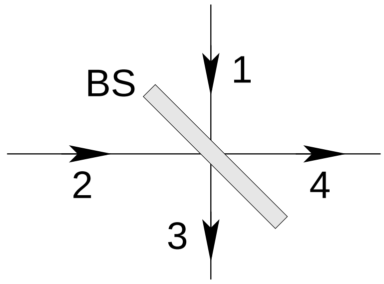

Figure 9. Sketch of a beam splitter. Modes (1,2) are transformed into $(3, 4)$.

omitted for simplicity). For a stochastic field, the product $\mathcal{E}^{(-)}\mathcal{E}^{(+)}$ has to be additionally averaged on the different realizations of the ensemble.
In a quantum treatment $\boldsymbol{\mathcal{E}}^{(+)}$ becomes a destruction operator (in the Heisenberg picture) and the response of the detector arises from transitions from the ground state of the atoms $|g\rangle$ in the photocathode to highly excited quasi–free states $|e\rangle$ by absorption of photons, see Fig. 8. Initially, we have for the combined system "atom plus EMF" $|i\rangle = |g, \{n\}\rangle$. The electrical dipole interaction $\hat{H}_{dip} = -e\hat{\boldsymbol{\mathcal{E}}}\mathbf{r}_{\text{atom}}$ induces transitions to final states $|f\rangle = |e, \{n'\}\rangle$. With the golden rule and summing over all possible final states, we have for the total transition rate (for single photon absorption), see Refs. [8–10],

$$\begin{aligned} \Gamma(t) &= \frac{2\pi}{\hbar^2} \sum_f \left| \langle f|\hat{H}_{dip}|i\rangle \right|^2 \delta(E_e + n'\hbar\omega - E_g - n\hbar\omega)\,, \\ &= \zeta \langle \hat{\mathcal{E}}^{(-)}(\mathbf{r}, t)\hat{\mathcal{E}}^{(+)}(\mathbf{r}, t)\rangle\,. \end{aligned} \qquad (64)$$

Implicitly, we shall assume a perfect photocathode (ζ =const.) with unit quantum efficiency so that each absorbed photon causes an atom in the phototube to emit an electron and register a single count during times $t, t + dt$. In first order dipol approximation absorption of more than one photon (per transition) does not occur. The number of photons which a counter records in a finite interval of time is described by a photon count distribution function $P_m(T)$, where m is the number of recorded photons. For details see, e.g. Loudon[2], Ref. [12] (Ch. 5.4) gives a short version.

6.2. BEAM SPLITTERS

A beam–splitter (BS) is an optical device that splits a beam of light into two. It is the key element of most optical interferometers. In a quantum mechanical description one has to take into account two incoming "wave beams" with fields $\mathcal{E}_1, \mathcal{E}_2$ even if one of them is the vacuum[32, 33]. These beams are transformed into a transmitted and a reflected output beam with fields $\mathcal{E}_3, \mathcal{E}_4$, respectively. For a symmetric dielectric beam splitter (and equal polarizations) the complex amplitudes transform according to

$$\begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \sqrt{1-R} & \mathrm{i}\sqrt{R} \\ \mathrm{i}\sqrt{R} & \sqrt{1-R} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{65}$$

R is the (intensity) reflection coefficent.
In Quantum Optics amplitudes a_j become operators $\hat{a}_j$ and (65) represents a unitary base transformation. The counterpart of "no incident wave" refers to the vacuum state of that mode rather than to "nothing". A beam splitter does not "split" photons, rather it conserves the number of photons and acts as a random selector which divides the incident flow of photons in a reflected and a transmitted one. As a consequence, the photon statistics of the reflected/transmitted beams correspond to that of the input beam after a random selection process has taken place.
In particular, we discuss the case of a dielectric 50:50 beam splitter[4]

$$\hat{a}_3 = \frac{1}{\sqrt{2}}\Big(\hat{a}_1 + \mathrm{i}\hat{a}_2\Big), \qquad \hat{a}_4 = \frac{1}{\sqrt{2}}\Big(\hat{a}_2 + \mathrm{i}\hat{a}_1\Big). \tag{66}$$

6.2.1. *Single photon input in port 2, (vacuum in 1)*

$$|0_1, 1_2\rangle = \hat{a}_2^\dagger|0\rangle \longrightarrow \frac{1}{\sqrt{2}}\Big(|0_3, 1_4\rangle + \mathrm{i}|1_3, 0_4\rangle\Big).$$

Indices label different modes. Note, the output state is *entangled*, although the input was not.

6.2.2. *N photons in port 2, (vacuum in 1)*

$$|0_1, N_2\rangle = \frac{1}{\sqrt{N!}}\,(\hat{a}_2^\dagger)^N|0\rangle \longrightarrow \sum_{k=0}^{N} \mathrm{i}^k \sqrt{\frac{1}{2^N}\binom{N}{k}}\;|k\rangle_3\;|(N-n)\rangle_4.$$

The photon statistics in ports 3,4 is binominal.
Special case $N = 2$:

$$|0_1, 1_2\rangle \longrightarrow \frac{1}{2}|0_3, 2_4\rangle + \frac{1}{\sqrt{2}}|1_3, 1_4\rangle - \frac{1}{2}|2_3, 0_4\rangle.$$

The probabilty that both photons "go together" after passing the beam-splitter is $(1/2)^2 + (1/2)^2 = 1/2$.

6.2.3. *Single photons in ports 1 and 2*

$$|1_1, 1_2\rangle = \hat{a}_1^\dagger\hat{a}_2^\dagger|0\rangle \longrightarrow \frac{\mathrm{i}}{\sqrt{2}}\Big(|0_3, 2_4\rangle + |2_3, 0_4\rangle\Big).$$

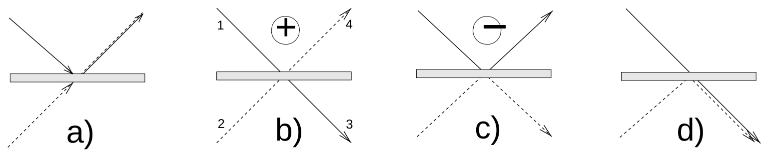

Figure 10. The four amplitudes of reflection and transmission of two photons at a 50:50 beam–splitter. Note the destructive interference of b) and c).

The output state is entangled and both photons "go together", see also Fig. 10. [N.B.: Entanglement is not a base invariant property.]

6.2.4. *α–state in port 2 (vacuum in 1)*

$$|0\rangle_1|\alpha\rangle_2 \longrightarrow \Big|\frac{\mathrm{i}\alpha}{\sqrt{2}}\Big\rangle_3 \Big|\frac{\alpha}{\sqrt{2}}\Big\rangle_4 .$$

As expected from classical electrodynamics the incident beam is splitted in two beams of half of the input intensity and there is a phase shift of $\pi/2$. [Hint: use (49)].
For a coherent state (with a Poissonian photon distribution), a random selection yields again a Poissonian, hence, a coherent state remains a coherent state after reflection or transmission by a beam splitter, yet with a reduced value of α. Moreover, the output state is not entangled (product of states in ports 3 and 4), i.e. the outputs are statistically independent.

6.2.5. *Thermal state in port 2 (vacuum in 1)*
A thermal state transforms under a beam splitter also in thermal states at the output ports. This nontrivial result may be conveniently obtained from the Glauber P–representation (53) and the result of Sect. 6.2.4.

$$\hat{\rho}_{in} = \Big(|0\rangle\langle 0|\Big)_1 \otimes \frac{1}{\pi\bar{n}} \int e^{-|\alpha|^2/\bar{n}} \Big(|\alpha\rangle\langle\alpha|\Big)_2 \, d^2\alpha \longrightarrow \hat{\rho}_{out}.$$

$\bar{n}$ denotes the mean input photon number. The reduced density operator for port 4

$$\hat{\rho}_4 = \mathrm{tr}_3 \hat{\rho}_{out} = \sum_n {}_3\langle n|\hat{\rho}_{out}|n\rangle_3$$

is also a thermal state (60), but with half of the mean photon number $\bar{n}_4 = \bar{n}/2$. The same holds for $\hat{\rho}_3$. Although the output state of the whole system is entangled, $\hat{\rho}_{out} \neq \hat{\rho}_3\,\hat{\rho}_4$, the (intensity) correlations between the photons in ports 3,4 are not affected. We leave this as problem P2.

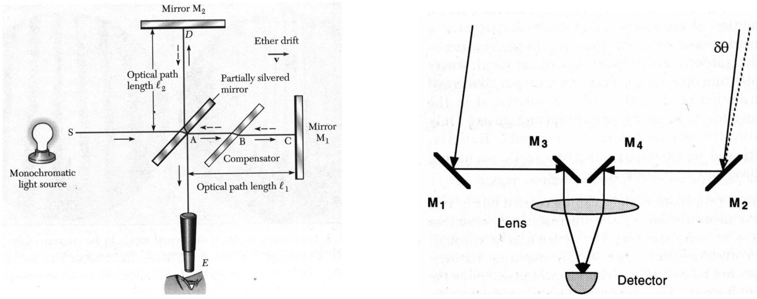

Figure 11. Sketch of the Michelson interferometer (left) and Michelson stellar interferometer (right). With these instruments the temporal and spatial correlation of the EMF can be measured independently. According to Bachor[6].

6.3. INTERFEROMETERS

Interferometers are devices to measure the correlation of the EMF between different space–time points $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$ by superposition of the (electrical) fields "on a screen" $(\mathbf{r}_s,\ t_s)$ by using pinholes/slits, mirrors, beam splitters and delay lines.

$$\mathcal{E}_{int}(\mathbf{r}_s, t_s) \propto \mathcal{E}(\mathbf{r}_1, t_1) + \mathcal{E}(\mathbf{r}_2, t_2).$$

The Michelson interferometers – as depicted in Fig. 11 – are particularily well suited to investigate coherence properties as these instruments separately measure the temporal and spatial dependencies of $G_1(2,1)$.
The (cycle averaged) intensity of light at position of the interferometer screen can be expressed in terms of the (first order) correlation function $G_1(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1)$

$$I(t) \propto G_1(1,1) + G_1(2,2) + 2\mathrm{Re}G_1(2,1), \tag{67}$$

$$G_1(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1) = \langle \hat{\boldsymbol{\mathcal{E}}}^{(-)}(\mathbf{r}_2, t_2) \hat{\boldsymbol{\mathcal{E}}}^{(+)}(\mathbf{r}_1, t_1) \rangle, \tag{68}$$

where $G_1(2,1)$ is short for $G_1(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1)$ etc. It is seen from (67) that the intensity on the screen consists of three contributions: The first two terms represent the intensities caused by each of the two path (or "slits") in the absence of the other, whereas the third term gives rise to interference effects. For a symmetric configuration (equal slit width, homogeneous illumination, $G_1(1,1) = G_1(2,2)$)) the visibility of the fringes is given by the magnitude of the normalized correlation function $g_1(1,2)$

$$\mathcal{V} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = |g(2,1)|, \tag{69}$$

$$g_1(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1) = \frac{G_1(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1)}{\sqrt{G_1(1,1) G_1(2,2)}}. \tag{70}$$

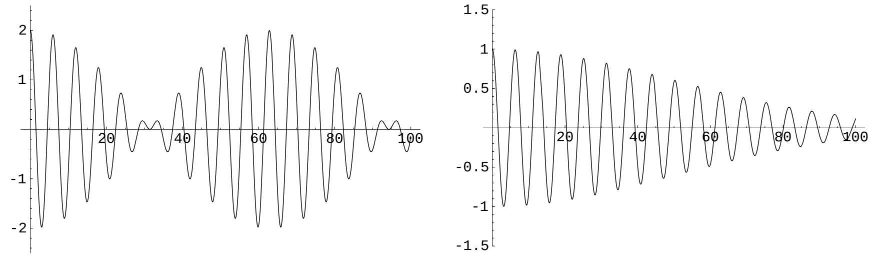

Figure 12. Real part of the coherence functions G_1 as a function of time delay. Left: two modes with $\omega_j = (1 \pm 0.05)\omega_0$ ("Na–D doublet"), right: many uncorrelated modes with a Gaussian spectrum.

Hence, coherence, i.e. the possibility of interference, of light is a measure of correlations in the EMF.

6.3.1. *Examples for G_1*

We study two examples for the (first order) classical coherence functions.

a) Single mode (deterministic/stochastic) field.

$$G_1(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1) = |\mathcal{E}^{(0)}_{\mathbf{k}}|^2 \langle |a_{\mathbf{k},\sigma}|^2 \rangle \exp\left(-\mathrm{i}\left[\mathbf{k}(\mathbf{r}_2 - \mathbf{r}_1) - \omega_{\mathbf{k}}(t_2 - t_1)\right]\right). \quad (71)$$

Obviously, this correlation function is (apart from the numerical value of $\langle |a_{\mathbf{k},\sigma}|^2 \rangle$) the same for a deterministic and a stochastic single mode field and displays maximum contrast for arbitrary space–time separations (unlimited spatial and temporal coherence).

b) Many statistically independent modes (of equal polarization) and intensity profile $I(\omega)$: $|\mathcal{E}^{(0)}(\omega_{\mathbf{k}})|^2 \langle \hat{a}^*_{\mathbf{k}',\sigma'}\, a_{\mathbf{k},\sigma} \rangle = I(\omega_{\mathbf{k}}) \delta_{\mathbf{k},\mathbf{k}'} \delta_{\sigma,\sigma'}$.

$$\mathcal{E}^{(+)}(\mathbf{r}, t) = \sum_{\mathbf{k}} \mathcal{E}^{(0)}_{\mathbf{k}} a_{\mathbf{k},\sigma}\, e^{\mathrm{i}(\mathbf{k}\mathbf{r} - \omega_{\mathbf{k}} t)}, \quad (72)$$

$$G_1(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1) = \sum_{\mathbf{k}} I(\omega_{\mathbf{k}})\, e^{-\mathrm{i}[\mathbf{k}(\mathbf{r}_2 - \mathbf{r}_1) - \omega_{\mathbf{k}}(t_2 - t_1)]}. \quad (73)$$

In particular, we have at the same space point, $\mathbf{r}_2 = \mathbf{r}_1 = \mathbf{r}$, (as measured by a Michelson interferometer)

$$G_1(t_2 - t_1) = G_1(\mathbf{r}, t_2; \mathbf{r}, t_1) = \int_0^\infty I(\omega)\, e^{\mathrm{i}\omega(t_2 - t_1)} \frac{d\omega}{2\pi} \quad (74)$$

For examples see Fig. 12.

For Gaussian and Lorentzian line–sources centered at $\omega_0 (> 0)$, we have

$$I(\omega) = \begin{cases} I_0\, e^{-\frac{(\omega - \omega_0)^2}{2(\Delta\omega)^2}}, \\ I_0\, \frac{(\Delta\omega)^2}{(\omega - \omega_0)^2 + (\Delta\omega)^2}, \end{cases} \qquad g_1(t) = \begin{cases} e^{-\frac{(\Delta\omega t)^2}{2}}\, e^{\mathrm{i}\omega_0 t}, & \text{(a)} \\ e^{-\Delta\omega |t|}\, e^{\mathrm{i}\omega_0 t}. & \text{(b)} \end{cases} \quad (75)$$

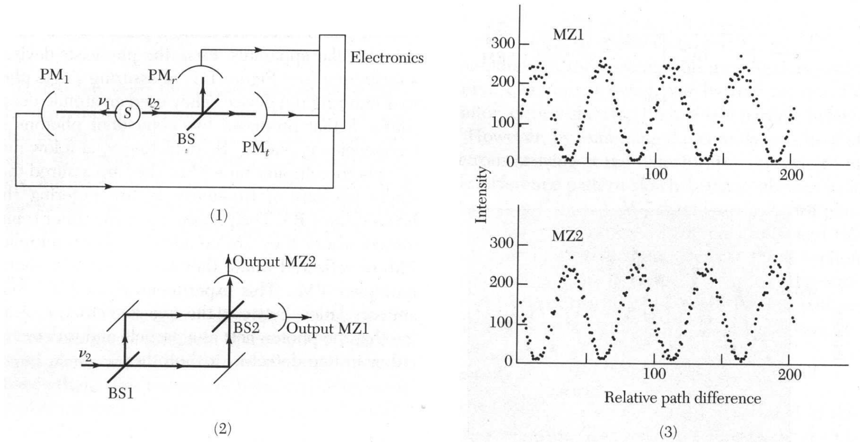

Figure 13. (1) Triggered photon cascade experiment to produce single photon states. (2) Mach–Zehnder Interferometer. (3) Number counts in the outputs of the photodetector MZ1 and MZ2 as a function of the path difference between the arms of the interferometer. (One channel corresponds to a variation of $\delta = \lambda/50$). According to Grangier et al.[34].

<u>Wiener–Khinchine theorem:</u>

- $G_1(\mathbf{r}, t)$ is the Fourier–transform of the intensity spectrum $I(\omega)$.
- Concerning temporal coherence, the filtered many mode field is virtually indistinguishable from the single mode field provided the time difference is less than the coherence time, $t < t_{coh} \sim 1/\Delta\omega$. The same reasoning holds for filtering within various directions in $\mathbf{k}$–space by apertures (spatial coherence).

6.3.2. *Historical Remarks*

Prior to the invention of the laser, coherent light was made out of "chaotic" radiation by (wave length) filters and apertures. Here, the visibility in the Michelson interferometers vanishes (and remains zero!) if the distance between the arms becomes larger than the (longitudinal) coherence length $l_{coh} = ct_{coh}$ (point like source of spectral width $\Delta\lambda$) or larger than the (transversal) coherence diameter d_{coh} (monochromatic source of angular diameter $\Delta\theta$)

Coherence time: $\quad t_{coh} = \frac{2\pi}{\Delta\omega} = \frac{\lambda_0^2}{c\Delta\lambda}$

Coherence diameter: $\quad d_{coh} = \frac{2\pi}{\Delta k} = 1.22\frac{\lambda}{\Delta\theta}$

The numerical factor of 1.22 holds for a circular light source. Note, the vanishing of the interference pattern is not the result of interference but of

the shift of individual patterns which are produced independently by each frequency component or each volume element of an extended source.
Examples:

red Cd–line:	$\lambda_0 = 643.8\text{nm}$,	$\Delta\lambda = 0.0013\text{nm}$,	$l_{coh} = 32\text{cm}$.
sun:	$\lambda_0 \sim 500\text{nm}$,	$\Delta\theta = 32'$,	$d_{coh} = 66\mu\text{m}$.

Since 1985, coherence experiments with genuine single photon states are possible, see Fig. 13. A single (!) Ca–atom is excited by a two–photon transition which, subsequently, decays by emitting two photons consecutively in opposite directions. One photon is used as a trigger, the second photon enters the Mach–Zehnder interferometer (MZI). Although beam splitters may change the photon correlations, first order coherence properties are not affected.

7. Photon Correlations

7.1. THE HANBURY BROWN & TWISS EFFECT

The idea to use intensity correlations (in thermal radiation) instead of field correlations [e.g. using a Michelson interferometer] dates back to 1949 where Hanbury Brown and Twiss, two radio astronomers at Jodrell Bank (Great Britain), were trying to design a radio interferometer which would solve the intriguing problem of measuring the angular size of the most prominent radio sources at this time: Cygnus A and Cassiopeia A. If, as some people thought their angular size is as small as the largest visible stars, then a global base line would be needed and a coherent superposition of the radio signals have been impossible in praxis those days.
First, a pilot model was built in 1950 and was tested by measuring the angular diameter of the sun at 2.4m wavelength, and, subsequently, the radio sources Cygnus A and Cassiopeia A. The intermediate–frequency outputs of the two independent superheterodyne receivers were rectified in square law detectors and bandpass filtered ($1 \ldots 2.5$KHz). Then, the low frequency (LF) outputs were brought together (by cable, radio link or telephone) and after analogue multiplication and integration the correlator output

$$G_2^{cl}(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1) = \langle I(\mathbf{r}_2, t_2)\, I(\mathbf{r}_1, t_1)\rangle \tag{76}$$

was investigated as a function of antenna separation $|\mathbf{r}_2 - \mathbf{r}_1|$.
For many independent modes, Eq. (76) can be evaluated in the same way as for G_1. As a result, we have (omitting $\mathcal{E}_{\mathbf{k}}^{(0)}$)

$$\begin{aligned} G_2^{cl}(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1) \;&=\; \sum_{\mathbf{k}} |\mathcal{E}_{\mathbf{k}}^{(0)}|^4 \left(\langle |a_{\mathbf{k}}|^4\rangle - 2\langle |a_{\mathbf{k}}|^2\rangle^2\right) + \\ &+\;\; |G_1(0,0;0,0)|^2 + |G_1(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1)|^2. \end{aligned} \tag{77}$$

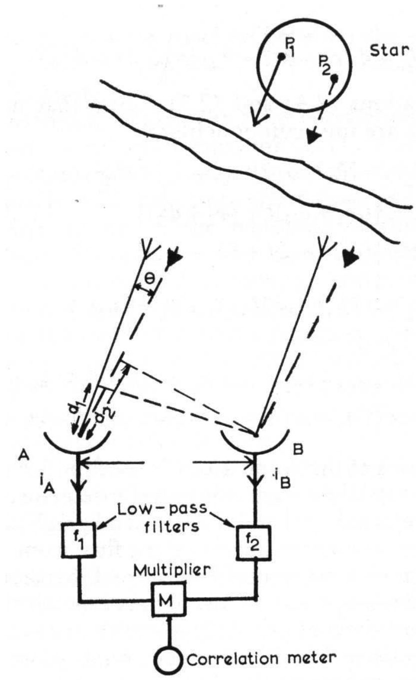

Figure 14. Optical intensity interferometer proposed and built in Australia by Hanbury Brown[35] to measure the angular diameter of stars. (Today there are much better methods/instruments available).

Moreover, for thermal radiation (Gaussian statistics) $\langle|a_{\mathbf{k}}|^4\rangle = 2\langle|a_{\mathbf{k}}|^2\rangle^2$, so that the first term of Eq. (77) drops out. In all other cases this contribution is negative so that the "contrast" in $G_2(2,1)$ between adjacent $(\mathbf{r}_2, t_2)$, $(\mathbf{r}_1, t_1)$ is reduced. For thermal radiation, intensity correlation measurements yield the same information as conventional first order coherence experiments, e.g. using the Michelson interferometers. This was directly confirmed by experiment[35].

The optical analogue of the intensity interferometer seemed to be straightforward. Instead of two RF receivers, one uses two photodetectors and a correlator (or coincidence counter) measures the combined absorption of photons at different space time points $(\mathbf{r}_2, t_2)$ and $(\mathbf{r}_1, t_1)$, see Fig. 14. If one thinks in terms of photons one must accept that thermal photons at two well separated detectors are correlated – they tend to "arrive" in pairs ("photon bunching"). But how, if the photons are emitted at random in a thermal source, can they appear in pairs at two well separated detectors? First, there was a strong opposition from theory but eventually, this problem was settled by experiment which clearly showed photon bunching in thermal radiation[24, 36]. However, due to the low bandwidth of the electronics and

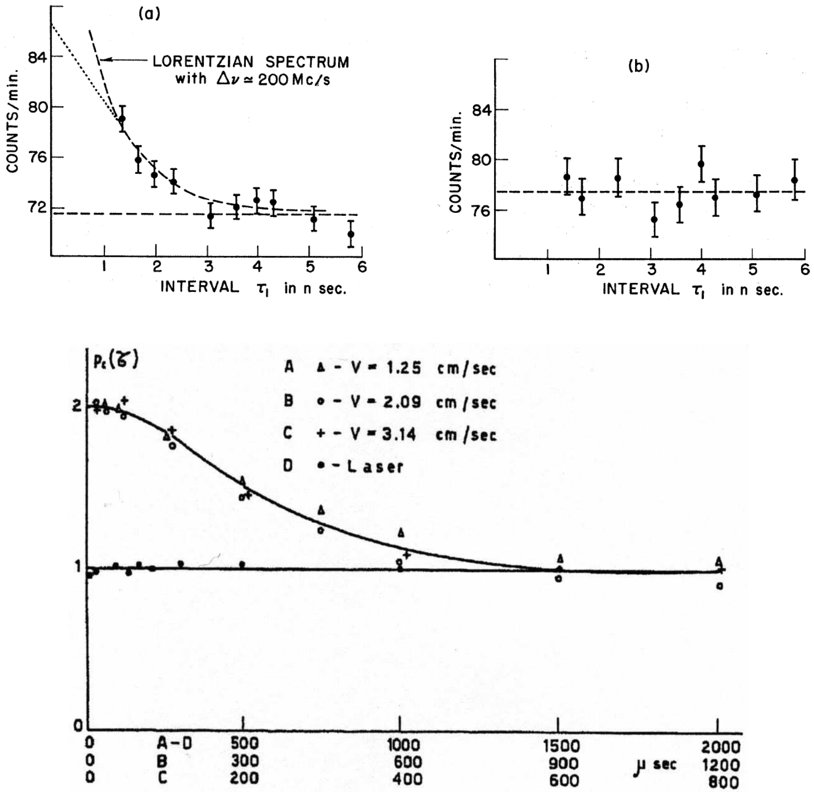

Figure 15. Measured photon coincidences in thermal and laser light. Upper panel: (a) mercury arc (Hg-198) and (b) tungsten lamp[37]. Lower panel: single mode laser light and chaotic light with ajustable coherence time[39].

low efficiency of the photocathodes the magnitude of the effect was only about 2% of the theoretical limit.

7.2. QUANTUM THEORY & EXPERIMENTS OF THE HBT EFFECT

In quantum theory, the classical result (77) is replaced by

$$G_2(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1) = \langle \hat{\mathcal{E}}^{(-)}(\mathbf{r}_2, t_2)\hat{\mathcal{E}}^{(-)}(\mathbf{r}_1, t_1)\hat{\mathcal{E}}^{(+)}(\mathbf{r}_2, t_2)\hat{\mathcal{E}}^{(+)}(\mathbf{r}_1, t_1)\rangle. \quad (78)$$

Here, operators are in the *Heisenberg–picture* and $\hat{\mathcal{E}}^{(+)}(\mathbf{r}, t) \propto \hat{a}\ \exp(-\mathrm{i}\omega t)$ denotes the positive frequency part of the electrical field. In contrast to the classical formulation, the sequence of operators is different from (77), where $I \propto \hat{\mathcal{E}}^{(-)}\hat{\mathcal{E}}^{(+)}$ – creation and destruction operators are in "normal order" (creation operators left to the destruction operators). Nevertheless, for thermal radiation, the classical result (77) remains valid.

Today, the photon bunching effect can be simply demonstrated with an artificial "chaotic" source which synthesizes pseudothermal light by passing laser radiation through a rotating ground glass disk (speed v) with long adjustable coherence times ("Martienssen lamp")[38], see Fig. 15.
It is instructive to define a coincidence ratio

$$R = \frac{C - C_{rand}}{C_{rand}} = \frac{(\Delta N)^2 - < \hat{N} >}{(< \hat{N} >)^2},$$

where $C = G_2(1,1) =< \hat{N}(\hat{N}-1) >$. The number of random coincidences is proportional to $C_{rand} = G_2(2,1) =< \hat{N} >^2$ when the separation of $\mathbf{r}_2, t_2$, $\mathbf{r}_1, t_1$ is much larger than the coherence area/time.

α–states:	$(\Delta N)^2 = \bar{n}$,	$R = 0$,
thermal states:	$(\Delta N)^2 = \bar{n}(\bar{n}+1)$,	$R = 1$,
number states:	$(\Delta N)^2 = 0$.	$R = -\frac{1}{n}$.

Classical states have photon number distributions which are always broader than a Poissonian, i.e., $(\Delta N)^2 \geq \bar{n}$, hence, the correlation ratio is always positive, $R \geq 0$, ("photon bunching"). α states (as generated by an amplitude stabilized laser) represent the optimum with respect to low photon fluctuations.
On the other hand, states which have less photon number fluctuations than a Poissonian, e.g., the number states, show "antibunching", i.e., the photons prefer to "come not too close". In particular, the single photon state $|n = 1\rangle$ is the most nonclassical state one can think of! Obviously, photon bunching is not "a typical Bose property".
The generation of nonclassical light (which still showed "photon bunching") was first demonstrated in 1977 by Mandel's group[40] whereas the first clear evidence for anti–bunching was presented by Diedrich and Walther[41] in 1987, using a single Mg–Ion in a Paul–trap.
As an application of photon correlations we mention the determination of the diffusion coefficient of protein molecules in aqueous solution, see Fig. 16. The diffusion constant D controls the Brownian motion which is characterized by the density–fluctuation autocorrelation function

$$C(\mathbf{r},t) = \langle \delta C(\mathbf{r}_2,t_2)\; \delta C(\mathbf{r}_1,t_1)\rangle = \langle C\rangle \frac{1}{(4\pi D t)^{3/2}} \exp\Big(-\frac{r^2}{4Dt}\Big).$$

$\langle C\rangle$ is the mean concentration and $r = |\mathbf{r}_2 - \mathbf{r}_1|$, $t = |t_2 - t_1|$. The light scattered in a fixed direction picks out the Fourier component $\mathbf{q} = \mathbf{k}_{\rm in} - \mathbf{k}_{\rm out}$ so that the first and second order coherence functions are

$$\begin{aligned} g_1(\mathbf{r},t) &= C(\mathbf{q},t)\, e^{-\mathrm{i}(\mathbf{k}_{\rm out}\mathbf{r}-\omega t)}, \quad C(\mathbf{q},t) = e^{-Dq^2 t} \\ g_2(\mathbf{r},t) &= 1 + f\,|g_1(\mathbf{r},t)|^2 = 1 + f\, e^{-2Dq^2 t}. \end{aligned}$$

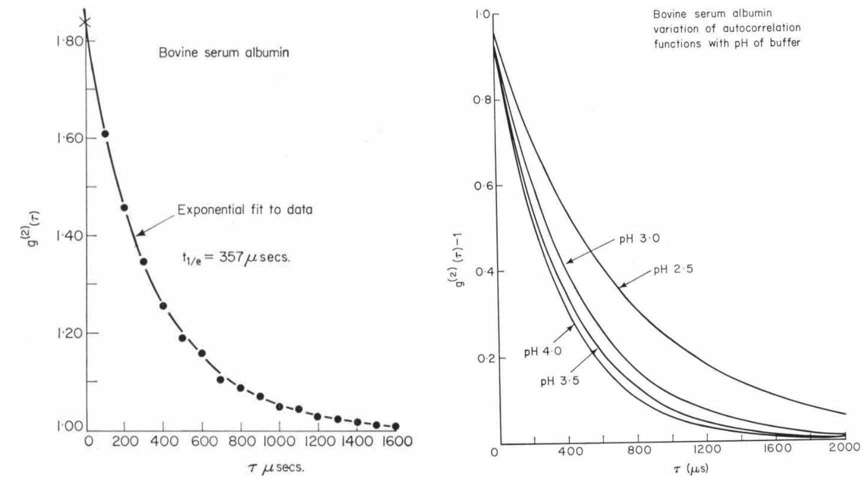

Figure 16. Left: Second order correlation function of laser light scattered from the protein bovine serum albumin. Right: effect of changing ph on the diffusion constant (and hence size of the protein). According to Pike, in Ref.[10].

$C(\mathbf{q}, t)$ is the Fourier–transform of $C(\mathbf{r}, t)$ and f $(0 < f \leq 1)$ is a "fudge factor" which depends on the finite area of the photocathode, finite counting time – see problem P3.

Today, photon correlation spectroscopy has become a versatile tool for investigating the spectral dynamics of single molecules which will be treated in more detail in lectures by Profs. Schwille and Webb during this school. Ultraweak correlated photon emission phenomena originating from biological organisms have been reported from time to time, e.g. see the recent article by Kobayashi and Inaba[42]. However, these phenomena resemble the pathological "mitogenic rays" reported by Gurwitsch[43] in 1923 ...

7.3. TWO PHOTON INTERFERENCE

How to measure the time interval between two photons? Do two different photons interfere? For intense laser pulses containing many photons this problem can be experimentally investigated using nonlinear optics. For single photons, this question was raised and answered by Mandel and coworkers more than 20 years ago[44, 45]. An outline of their experiment is shown in Fig. 17. A coherent beam of UV light with frequency ω_0 is parametrically down converted in a KDP crystal to a "signal" and an "idler" photon with $\omega_1 + \omega_2 = \omega_0$. Then, the two photons are directed by mirrors M_1, M_2 to pass a beam splitter BS, and the supposed beams interfere and are directed to detectors D_1, D_2. Neglecting the band–width of filters IF_1 and IF_2, the fields at detectors D_1, D_1 are (in the notation of Sect. 6.2.3 $1' = 3$, $2' = 4$)

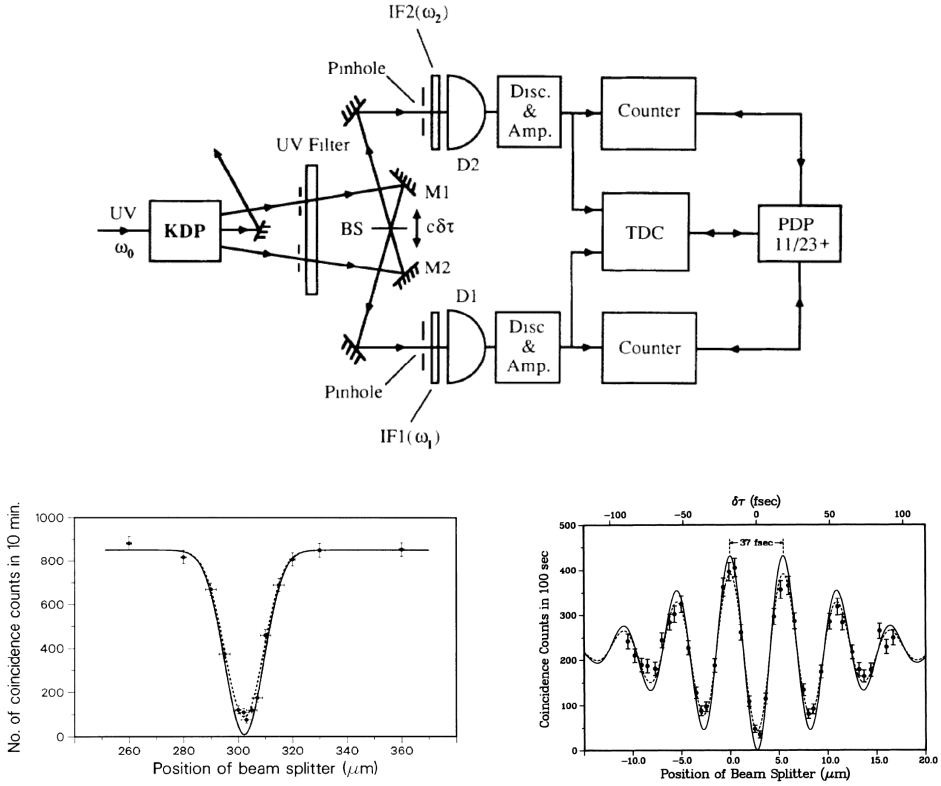

Figure 17. Interference of two photons as a function beam splitter displacement (time delay). Upper panel: Sketch of experimental set up; Lower panel: (left) both photons have the same frequency; (right): different frequencies. According to Mandel et al.[44, 45].

$$
\begin{aligned}
\hat{\mathcal{E}}^{(+)}_{1'}(t) &= \frac{1}{\sqrt{2}}\hat{a}_1\, e^{-\mathrm{i}\omega_1(t-\tau_1)} + \frac{\mathrm{i}}{\sqrt{2}}\hat{a}_2\, e^{-\mathrm{i}\omega_1(t-\tau_1+\delta\tau)}, \\
\hat{\mathcal{E}}^{(+)}_{2'}(t) &= \frac{1}{\sqrt{2}}\hat{a}_2\, e^{-\mathrm{i}\omega_2(t-\tau_1)} + \frac{\mathrm{i}}{\sqrt{2}}\hat{a}_1\, e^{-\mathrm{i}\omega_2(t-\tau_1+\delta\tau)}.
\end{aligned}
$$

τ_1 is the propagation time between mirror M_1 and detector D_1 (or M_2, D_2). According to (78) the joint probability of photodetection by D_1 at time t and D_2 at $t+\tau$ is

$$
\begin{aligned}
P^{(0)}_{12}(\tau) &\propto \langle \hat{\mathcal{E}}^{(-)}_{1'}(t)\hat{\mathcal{E}}^{(-)}_{2'}(t+\tau)\hat{\mathcal{E}}^{(+)}_{2'}(t+\tau)\hat{\mathcal{E}}^{(+)}_{1'}(t)\rangle, \\
&\propto 1-\cos[(\omega_2-\omega_1)\delta\tau] \;\propto\; \sin^2\left[\frac{(\omega_2-\omega_1)\delta\tau}{2}\right].
\end{aligned}
$$

(Beware of $\tau \neq \tau_1$). The rate at which photons are detected in coincidence, when BS is displaced from its symmetry position by $\pm c\delta\tau$ displayes a typical interference pattern, see Fig. 17. Nevertheless, signal and idler photons have no definite phase, and are therefore mutually incoherent, in the sense that the individual signals from D_1 and D_2 exhibit no oscillatory structure. In praxis, the spectral distributions of signal and idler photons, band–widths of filters IF_1 and IF_2 as well as the resolving time of the detectors (i.e. integration on τ) must be taken into account which can be lumped into a coherence factor f_c.

$$P_{12} \propto 1 - f_c \cos[(\omega_1 - \omega_2)\delta\tau] \,, \quad f_c = e^{-(\sigma\delta\tau)^2/2}.$$

Parameters:
a) $\omega_1 = \omega_2 = 2.77 \times 10^{15}\mathrm{s}^{-1}$: $\lambda = 680\mathrm{nm}$, $\Delta\omega = 3 \times 10^{13}\mathrm{s}^{-1}$, $\sigma = \sqrt{2}\Delta\omega$.
b) $\omega_1 \neq \omega_2$: mean wave length length $\bar{\lambda} = 680\mathrm{nm}$, ($\bar{\omega} = 2.77 \times 10^{15}\mathrm{s}^{-1}$).
$\omega_1 - \omega_2 = 1.70 \times 10^{14}\mathrm{s}^{-1}$, $\sigma = 1.85 \times 10^{13}\mathrm{s}^{-1}$. (No spectral overlap of IF_1 and IF_2. Coherence time $\approx$ 100fs.)
It is noteworthy that this interference technique, in principle, allows beating at optical frequencies $|\omega_2 - \omega_1|$ to be detected with photodetectors whose response times are thousands of times slower. In addition, both photons may even originate from different sources[46].

Problem P4:
Finally, one may ask the question: is the Hong–Ou–Mandel–effect a quantum effect or can it be understood classically?, i.e. will a "classical" state $|\mathrm{in}\rangle = |\alpha\rangle_1|\alpha\rangle_2$ lead to the same result as for two single photons, $|\mathrm{in}\rangle = |1\rangle_1|1\rangle_2$?

8. Outlook

Probably many of you are disappointed with the rather formal definition of photons given in Sect. 4.5. However, even Haroche and Raimond's superb book[7] *Exploring the Quantum* is less definitive. Therefore, let the master himself have the last words:

> *Die ganzen Jahre bewusster Grübelei haben mich der Antwort der Frage "Was sind Lichtquanten" nicht näher gebracht. Heute glaubt zwar jeder Lump, er wisse es, aber er täuscht sich...*
>
> Literal translation:
>
> *All the years of willful pondering have not brought me any closer to the answer to the question "what are light–quanta". Today every good–for–nothing believes he should know it, but he is mistaken...*

Albert Einstein
In a letter to M. Besso, 1951.

9. Acknowledgement

Due to unexpected circumstances, this contribution could not be presented at the summerschool. The author thanks Prof. Di Bartolo for the possibility to include this article in the proceedings nevertheless.

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Solutions to problems

1a) In Coulomb–gauge $\mathcal{E} = -\text{grad}\Phi$, $\mathcal{A} = 0$. The scalar potential is not quantized, hence, there are no photons in a static electric field.
1b) A static $\mathcal{B}$ field originates from a time independent current source, i.e. $f(t) = f = $const in (37) for a single mode $(\mathbf{k}, \sigma)$. Eigenstates of (58) can be found by the canonical transformation

$$\begin{aligned}
\hat{a} &= \hat{b} + f/\hbar\omega_0\,, \quad [\hat{b}, \hat{b}^\dagger] = 1\,, \\
\hat{H} &= \hbar\omega_0 \hat{a}^\dagger \hat{a} - \Big(f\,\hat{a}^\dagger + f^*\,\hat{a}\Big) = \hbar\omega_0 \hat{b}^\dagger \hat{b} + \frac{|f|^2}{\hbar\omega_0}, \\
\hat{N} &= \hat{a}^\dagger \hat{a} = \hat{b}^\dagger \hat{b} + \Big(\frac{f^*}{\hbar\omega_0}\hat{b} + hc\Big) + \left|\frac{f}{\hbar\omega_0}\right|^2 .
\end{aligned}$$

The expectation value of $\hat{N}$ in the ground state, $\hat{b}|g\rangle = 0$, is nonzero:

$$< \hat{N} >= \left|\frac{f}{\hbar\omega_0}\right|^2 > 0.$$

Hence, there are photons in a static magnetic field – but they are "virtual" and cannot "fly away"! [Result should be summed over all contributing modes].

2) Use the number representation of $|\alpha\rangle$ and polar "coordinates", $d^2\alpha = |\alpha|\, d|\alpha|\, d\phi$.

3) In a quasi–monochromatic approximation the scattered field is

$$\mathcal{E}^{(+)}(\mathbf{r}, t) = F(\mathbf{q}, t)\, \exp(\mathrm{i}[\mathbf{k}_{\text{out}}\mathbf{r} - \omega t]),$$

where F is the scattering amplitude. In Born–approximation

$$F(\mathbf{q}, t) \sim \int \delta C(\mathbf{r}', t) \exp(\mathrm{i}\mathbf{q}\mathbf{r}')\, d^3\mathbf{r}'.$$

G_1 follows from (68), $C(\mathbf{q}, t)$ is the (spatial) Fourier–transform of $C(\mathbf{r}, t)$,

$$G_1(\mathbf{r}, t) \sim C(\mathbf{q}, t)\, \exp[-\mathrm{i}(\mathbf{k}_{\text{out}}\mathbf{r} - \omega t)].$$

As $C(\mathbf{r}, t)$ is a solution of the diffusion equation, $C(\mathbf{q}, t) = \exp(-D\mathbf{q}^2 t)$,

$$\frac{\partial}{\partial t} C(\mathbf{r}, t) = D\Delta C(\mathbf{r}, t), \rightarrow \frac{d}{dt} C(\mathbf{q}, t) = -D\mathbf{q}^2 C(\mathbf{q}, t).$$

Moreover, diffusion is a gaussian process, hence G_2 is obtained from (77). Within the coherence area, G_2 is independent of $\mathbf{r}$.

4) $\hat{\mathcal{E}}_{2'}^{(+)}(t+\tau)\hat{\mathcal{E}}_{1'}^{(+)}(t)$ leads to operator products $\hat{a}_1\hat{a}_2, \hat{a}_1^2, \hat{a}_2^2$ which, upon operation on $|\alpha\rangle$, can be replaced by α^2, see (48). Analogous for $\langle\alpha|\mathcal{E}_{1'}^{(-)}\hat{\mathcal{E}}_{2'}^{(-)}$. Fast oscillating terms, e.g. $\cos^2[(\omega_1 + \omega_2)\delta\tau]$, can be replaced by their averages. This leads to an additional contribution of $1/2$ to $P_{12}^{(0)}(\tau)$. Hence, the visibility contrast (69) is reduced to $\mathcal{V} = 1/2$, whereas $\mathcal{V} = 1$ for two single photons in ports 1,2.

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